15.02.23

## Lecture-10

**Question-1** If $u = f(y-z, z-x, x-y)$ prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Let $X = y - z$

$Y = z - x$

$Z = x - y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial x}$$

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z} \quad \text{-------- 1.}$$

New,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} \quad \text{-------- 2.}$$

New,

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial z}$$

$$\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y} \quad \text{------ 3.}$$

on adding eq 1, 2 and 3,

$$= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z} + \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} - \frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y}$$

$$= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Hence Proved. Ans.

**Question-2** If $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$ show that $\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$

~~$\frac{\partial f}{\partial x} =$~~ New, $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial r}$.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y} \cdot \sin\theta \quad \text{------ 1.}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$\frac{\partial w}{\partial \theta} = -\frac{\partial w}{\partial x}\, r\sin\theta + \frac{\partial w}{\partial y}\, r\cos\theta$$

$$\frac{1}{r}\frac{\partial w}{\partial \theta} = -\sin\theta\,\frac{\partial w}{\partial x} + \cos\theta\,\frac{\partial w}{\partial y} \quad \text{------ 2.}$$

on squaring and adding eq. 1 and 2.

$$\Rightarrow \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial w}{\partial y}\right)^2\sin^2\theta + \cancel{2\sin\theta\cos\theta\,\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}} +$$

$$\left(\frac{\partial w}{\partial y}\right)^2\cos^2\theta + \left(\frac{\partial w}{\partial x}\right)^2\sin^2\theta - \cancel{2\sin\theta\cos\theta\,\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}}.$$

$$\Rightarrow \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \cos^2\theta\left(\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right) + \sin^2\theta\left(\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right)$$

$$= \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$

$$= \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

Hence Proved. Ans

Question-3 If $\phi(cx - az,\ cy - bz) = 0$ show that $a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = c$.

Let $X = cx - az$
$Y = cy - bz$. i.e. $\phi(X, Y) = 0$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial x}.$$

$$\frac{\partial z}{\partial x} = c\,\frac{\partial z}{\partial X} \quad \text{------ 1.}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial Y}$$

$$= \frac{\partial z}{\partial X}\times 0 + c\,\frac{\partial z}{\partial Y}$$