16·02·23

Lecture-11

Question-1 If $\phi(cx - az, cy - bz) = 0$, Show that
$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = c.$$
$\because$ z is a dependent variable

Let $u = cx - az$

$v = cy - bz.$

$\phi(u, v) = 0.$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial x}.$$

$$0 = \frac{\partial \phi}{\partial u}\left(c - a\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(-b\frac{\partial z}{\partial x}\right).$$

$$= c\frac{\partial \phi}{\partial u} - \frac{\partial z}{\partial x}\left(a\frac{\partial \phi}{\partial u} + b\frac{\partial \phi}{\partial v}\right)$$

$$\frac{\partial z}{\partial x} = \frac{c\,\partial\phi/\partial u}{a\frac{\partial \phi}{\partial u} + b\frac{\partial \phi}{\partial v}}.$$

$$a\frac{\partial z}{\partial x} = \frac{ac\;\partial\phi/\partial u}{a\frac{\partial \phi}{\partial u} + b\frac{\partial \phi}{\partial v}} \quad \text{---------- 1.}$$

Now,

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\left(-a\frac{\partial z}{\partial y}\right) + \frac{\partial \phi}{\partial v}\left(c - b\frac{\partial z}{\partial y}\right)$$

$$0 = c\frac{\partial \phi}{\partial \cancel{y}} - \frac{\partial z}{\partial y}\left(a\frac{\partial \phi}{\partial u} + b\frac{\partial \phi}{\partial v}\right)$$

$$\frac{\partial z}{\partial y} = \frac{c\,\partial\phi/\partial \cancel{y}v}{a\frac{\partial \phi}{\partial u} + b\frac{\partial \phi}{\partial v}}.$$

$$b\frac{\partial z}{\partial y} = \frac{bc\;\partial\phi/\partial \cancel{y}v}{a\frac{\partial \phi}{\partial u} + b\frac{\partial \phi}{\partial v}} \quad \text{---------- 2.}$$

on adding eq. 1. and 2.

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = \frac{ac\;\partial\phi/\partial u + bc\;\partial\phi/\partial v}{a\frac{\partial \phi}{\partial u} + b\frac{\partial \phi}{\partial v}}.$$

$$= \frac{a\,\partial z}{\partial x} + \frac{b\,\partial z}{\partial y} = \frac{c\left(a\,\partial\phi/\partial u + b\,\partial\phi/\partial v\right)}{\frac{a\,\partial\phi}{\partial u} + \frac{b\,\partial\phi}{\partial v}}.$$

$$\frac{a\,\partial z}{\partial x} + \frac{b\,\partial z}{\partial y} = c.$$

Hence Proved - Ans.

Question-2 If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

Let $v = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$

$w = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$.

$u = u(v, w)$.

Now,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x}.$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial w}\left(\frac{-1}{x^2}\right)$$

$$x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} \quad \text{------ 1.}$$

Now,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial y}.$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial y}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial w}(0)$$

$$y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} \quad \text{------ 2.}$$

Now,

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}(0) + \frac{\partial u}{\partial w}\left(\frac{1}{z^2}\right)$$

$$z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial w} \quad \text{------ 3.}$$

On adding 1, 2 and 3.

$$= x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} + \frac{\partial u}{\partial v} +$$

$$= x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0.$$

Hence Proved. Ans.