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Lecture-12

Limit And Continuity

Limits are of two types:-

1. Simultaneous form:- A limit of the type $\lim_{(x,y)\to(a,b)} f(x,y)$ is called simultaneous limit. It can also be written as $\lim_{\substack{x\to a\\ y\to b}} f(x,y)$

2. Iterated limit :- (Repeated limit)

The limit of the form $\lim_{x\to a}\lim_{y\to b} f(x,y)$; $\lim_{y\to b}\lim_{x\to a} f(x,y)$ are called iterated limits or repeated limit. These two limit may or may not be equal.

Note:- If the repeated limit are not equal then simultaneous limit cannot exist.

Question-1 Prove that the simultaneous limit

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$ will not exist.

Along the path $y = x$.

$$\lim_{x\to 0} \frac{x^4}{x^2+x^6}$$

$$= \lim_{x\to 0} \frac{x^2}{1+x^4}$$

$$= 0.$$

Now, Along the curve $y^3 = x$.

$$\lim_{x\to 0} \frac{x^2}{x^2+x^2}$$

$$= \lim_{x\to 0} \frac{1}{2} = 1/2.$$

$\because$ along different paths have different limiting value

$\therefore$ The simultaneous limit does not exist.

Hence Proved. Ans.

Continuity of function of two variables :-

The function $f(x, y)$ is said to be continuous at (a, b) if for every $\epsilon > 0$, $\exists\ \delta < 0$ s.t. the $|f(x, y) - f(a, b)| < \epsilon$, $\forall\ |x - a| < \delta$, $|y - b| < \delta$.

OR

A function $f(x, y)$ is said to be continuous at (a, b) if the simultaneous limit $\lim_{(x,y)\to(a,b)} f(x,y)$ exist and its equal to its functional value $f(a, b)$.

Question-2 Test the contnuity at $(1, 2)$ of the function

$$f(x, y) = \begin{cases} x^2 + 4y, & (x, y) \neq (1, 2) \\ 0, & (x, y) = (1, 2) \end{cases}$$

$$\lim_{(x,y)\to(1,2)} f(x, y) = \lim_{(x,y)\to 1,2} x^2 + 4y$$

$$= 1^2 + 8$$

$$= 9$$

$f(1, 2) = 0$

$\therefore$
$$\lim_{(x,y)\to(1,2)} f(x, y) \neq y(1, 2)$$

$\therefore$ the given function is not continious at point $(1, 2)$.

Question-3 If $f : R^2 \to R$

$$f(x, y) = \begin{cases} \dfrac{(y^2 - x^2)xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

check its contiunity at $(0, 0)$.

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{(y^2 - x^2)xy}{x^2 + y^2}$$

Along the path $y = mx$.

$$\lim_{x\to 0} \frac{x^2(m^2 - 1)mx^2}{(1 + m^2)x^2}$$

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$$= \lim_{x \to 0} \frac{x^2 (m^2 - 1) m}{1 + m^2}$$

Now,

$$= 0 , \forall m$$

$$f(0,0) = 0.$$

$\because$ $f(x, y)$ has its simultaneous limit equal to value of the function at $(0, 0)$.

$\therefore$ The given function is continuous. Ans.

Also, To check,

Along the path $x = my$,

$$= \lim_{y \to 0} \frac{y^2 (1 - m^2) m y^2}{(m^2 + 1) y^2}$$

$$= \lim_{y \to 0} \frac{y^2 (1 - m^2) m}{m^2 + 1}$$

$$= 0 , \forall m.$$

Hence confirmed. Ans.