20.02.23

Lecture - 13

Question-1 Test the continuity for the function

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$\lim_{(x,y) \to (0,0)} f(x, y) = \frac{x^3 - y^3}{x^2 + y^2}$$

Along the path $y = mx$

$$\lim_{x \to 0} \frac{x^3(1 - m^3)}{x^2(1 + m^2)}$$

$$\lim_{x \to 0} \frac{x(1 - m^3)}{1 + m^2}$$

$$= 0 \quad \forall m$$

Now, Along the path $x = my$.

$$= \lim_{y \to 0} \frac{y^3(m^3 - 1)}{y^2(m^2 + 1)}$$

$$= \lim_{y \to 0} \frac{y(m^3 - 1)}{m^2 + 1}$$

$$= 0 \quad \forall m$$

Now

$f(0, 0) = 0$

$\because$ $f(x, y)$ has its simultaneous limit equal to value of the function at $(0, 0)$

Hence, it is continuous. Ans.

Question-2 Discuss the continuity of

$$f(x, y) = \begin{cases} \dfrac{x}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0). \\ 2 & (x, y) = (0, 0). \end{cases}$$

$$= \lim_{(x,y) \to (0,0)} f(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$$

Along the path $y = mx$.

$$= \lim_{x \to 0} \frac{x}{\sqrt{m^2+1}\,(x)}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{1+m^2}}$$

$\because$ the simultaneous limit along the path $y = mx$ is not unique for different value of m, and limit does not exist.

$\therefore$ the given function is not continuous ~~at~~ Ans.

Question-3 Evaluate the following limit :

1. $$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y}{x^4 + y^2}$$

Along the $y = x^2$

$$= \lim_{x \to 0} \cancel{\frac{y^2}{y^2(1+}} \frac{x^4}{2x^4}$$

$$= \lim_{x \to 0} \frac{1}{2}$$

$$= 1/2.$$

Along the path $y = x$

$$\lim_{x \to 0} \frac{x^3}{x^4 + x^2}$$

$$= \lim_{x \to 0} \frac{x}{x^2 + 1}$$

$$= 0.$$

$\because$ the simultaneous limit are not equal.

$\therefore$ limit does not exist. Ans.

2. $$\lim_{\substack{x \to 0 \\ y \to 0}} x^3 + y^3.$$

Along the path $y = x$.

$$= \lim_{x \to 0} 2x^3$$

$= 0.$

Now,

Along the path $y = mx$.

$$= \lim_{x\to 0} x^3(1+m^3) = 0.$$

$\therefore$ the simultaneous limit exists. Ans.

3. $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2 - x^2}{y^2 + x^2}$$

Along the path $y = mx$

$$= \lim_{x\to 0} \frac{(m^2-1)x^2}{(m^2+1)x^2}$$

$$= \lim_{x\to 0} \frac{m^2-1}{m^2+1} = \frac{m^2-1}{m^2+1}$$

$\therefore$ the limits are not unique $\forall m$, therefore simultaneous limit does not exist. Ans

4. $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3 - y^3}{x^2 + y^2}$$

Along the path $y = mx$.

$$= \lim_{x\to 0} \frac{x^3(1-m^3)}{x^2(1+m^2)}$$

$$= \lim_{x\to 0} \frac{x(1-m^3)}{1+m^2}$$

$$= 0.$$

Along the path $x = my$.

$$= \lim_{y\to 0} \frac{y^3(m^3-1)}{y^2(m^2+1)}$$

$$= \lim_{y\to 0} \frac{y(m^3-1)}{m^2+1}$$

$$= 0.$$

$\therefore$ the simultaneous limits exist. Ans.