21·02·23

Lecture -14

Directional Derivative

If u is defined on $S \subseteq R^2$ and $c \in S$. Let $u = (u_1, u_2) \in R^2$ then the directional derivative of f is at c in the directional of u is defined,

$$f'(c, u) = \lim_{h \to 0} \frac{f(c+uh) - f(c)}{h}$$, provided this limit exists.

Question-1 Let $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$

find the directional derivative along $u = (\sqrt{2}, \sqrt{2})$ at $(0,0)$

Directional derivative of f at c in the direction of u i.e.

$$f'(c, u) = \lim_{h \to 0} \frac{f(c+uh) - f(c)}{h}$$

$$= \lim_{h \to 0} \frac{f(\sqrt{2}h, \sqrt{2}h) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2}\cancel{h} \cdot 2h^2}{\cancel{h}(2h^2 + 4h^4)}$$

$$= \lim_{h \to 0} \frac{\cancel{2}\sqrt{2}}{\cancel{2}(1+2h^2)}$$

$$= \lim_{h \to 0} \frac{\sqrt{2}}{1+2h^2}$$

$$= \sqrt{2}. \text{ Ans.}$$

Partial Derivative:-

Suppose $z = f(x,y)$ be a function of two independent variables x and y. Then the partial derivative of z with respect to x when y is treated as constant at (a,b) is defined

$$f_x(a,b) = \left(\frac{\partial f}{\partial x}\right)_{(a,b)} = \left(\frac{\partial z}{\partial x}\right)_{(a,b)} = \lim_{h \to 0} \frac{f(a+h, b) - f}{h}$$

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provided limit exists.
similarly,

$$f_y(a,b) = \left(\frac{\partial f}{\partial y}\right)_{(a,b)} = \left(\frac{\partial z}{\partial y}\right)_{(a,b)} = \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k}.$$

Note:- When we consider the function of two variables the existance of partial derivatives at a point need not imply continuity at that point.

Question-2. Calculate $f_x(0,0)$ and $f_y(0,0)$, if $f: R^2 \to R^2$ be function defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$= \lim_{h\to 0} \frac{(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h0}{0+h^2}$$

$$\lim_{h\to 0} \frac{0}{h}$$

$$= 0. \quad \text{Ans.}$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

$$= \lim_{k\to 0} \frac{\cancel{x}0h^2}{0+h^4}$$

$$= \lim_{k\to 0} \frac{0}{h^2}$$

$$= 0. \quad \text{Ans.}$$

Differentiability :-
If $f(x,y)$ is given then $\phi(h,k) = \frac{f(h,k)}{\sqrt{h^2+k^2}}$

If $\lim_{(h,k)\to(0,0)} \phi(h,k) = 0$ then f is differentiable otherwise no by choosing the path $k = mh$.

Question-3 suppose $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

then prove that f has partial derivative at $(0,0)$ but is not differentiable at the origin.

$\lim (h,k) \to (0,0)$ $\lim h,k) \to (0,0)$

$$\lim_{(h,k)\to(0,0)} \phi(h,k) = \frac{f(h,k)}{\sqrt{h^2+k^2}} = \frac{hk}{(h^2+k^2)}.$$

Along the path $k = mh$.

$$= \lim_{h\to0} \frac{mh^2}{h^2(1+m^2)}$$

$$= \lim_{h\to0} \frac{m}{1+m^2}$$

$$= \frac{m}{1+m^2}$$

→ which depend upon m.
Hence the simultaneous limit does not exist