

Jacobians :- If u and v are the functions of two independent variables x and y , then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian of u and v with respect to x and y . It is denoted by $J(u, v)$ or $\frac{\partial(u, v)}{\partial(x, y)}$.

$$J(u, v, w) = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties of Jacobian :- (Chain Rule).

1. If u, v are functions of r, s where r, s are functions of x, y then

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}$$

Proof :- $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} = u_r \cdot r_x + u_s \cdot s_x$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} = u_r \cdot r_y + u_s \cdot s_y$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial x} = v_r \cdot r_x + v_s \cdot s_x$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial y} = v_r \cdot r_y + v_s \cdot s_y$$

Now,

$$\text{RHS :- } \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} u_r v_x + u_s s_x & u_r v_y + u_s s_y \\ v_r v_x + v_s s_x & v_r v_y + v_s s_y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}$$

Ans

hence proved Ans.

2. If J_1 is the Jacobian of u, v with respect to x, y and J_2 is the Jacobian of x, y with respect to u, v then

$$J_1 J_2 = 1$$

Ans

$$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = 1.$$

Proof:-

Let $u = u(x, y)$
 $v = v(x, y)$

Differentiating v partially w.r.t. u and v , we get

$$1 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$1 = u_x x_u + u_y y_u$$

Now, differentiating partially w.r.t. v

$$0 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$0 = u_x x_v + u_y y_v$$

Now, differentiating v partially w.r.t. u we get

$$0 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$0 = v_x x_u + v_y y_u$$

$$1 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$I = v_x x_v + v_y y_v.$$

Now, LHS:-

$$\begin{aligned} &= \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \\ &= \begin{vmatrix} u_x x_u + u_y y_u & u_x x_v + u_y y_v \\ v_x x_u + v_y y_u & v_x x_v + v_y y_v \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ &= 1, \\ &\therefore \text{RHS.} \end{aligned}$$

Hence Proved. Ans.