Lecture-16

23·02·23

Theorem:- If the relation connecting u_i's and x_i's are of the form

$$u_1 = f_1(x_1)$$
$$u_2 = f_2(x_1, x_2)$$
$$\vdots$$
$$u_n = f_n(x_1, x_2 \ldots x_n).$$

then $$\frac{\partial(u_1, u_2 \ldots u_n)}{\partial(x_1, x_2 \ldots x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \ldots \frac{\partial u_n}{\partial x_n}$$

Theorem:- If functions u, v, w of three independent variables x, y, z are not independent then the Jacobian of u, v, w with respect to x, y, z vanishes.

Jacobian of Implicit functions:-

If u_1, u_2 and u_3 are the implicit functions of x_1, x_2, x_3 i.e.

$$F_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$
$$F_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0.$$
$$F_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$

Then $$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = (-1)^3 \frac{\dfrac{\partial(F_1, F_2, F_3)}{\partial(x_1, x_2, x_3)}}{\dfrac{\partial(F_1, F_2, F_3)}{\partial(u_1, u_2, u_3)}}.$$

Theorem:- Let $u_1, u_2 \ldots u_n$ be functions of $x_1, x_2 \ldots x_n$, then the necessary condition for the existence of a relation of the form $F(u_1, u_2 \ldots u_n) = 0$ is that the Jacobian $\dfrac{\partial(u_1, u_2 \ldots u_n)}{\partial(x_1, x_2 \ldots x_n)}$ should vanish identically

Question :- 1 If $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$. Also prove that $J_1 J_2 = 1$.

We have,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r. \quad \text{Ans.}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial \theta/\partial x & \partial \theta/\partial y \end{vmatrix}$$

Now, $r = \sqrt{x^2+y^2}$.

$\theta = \tan^{-1}(y/x)$.

$$= \begin{vmatrix} x/\sqrt{x^2+y^2} & y/\sqrt{x^2+y^2} \\ -y/x^2+y^2 & x/\sqrt{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{x^2+y^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}. \quad \text{Ans.}$$

Now,

To prove that $J_1 J_2 = 1$.

LHS :- $= \frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)}$

$= r \cdot \frac{1}{r}$

$= 1$

:- RHS

Hence Proved. Ans.