24.2.23

Qus: If $u = xyz$, $v = xy + yz + zx$, $w = x+y+z$. Then compute the Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} yz & xz & xy \\ y+z & x+z & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \cancel{yz(x+z-x-y) - (y+z)[xz(x+y) - xy(x+}$$
$$+ 1($$

$\Rightarrow yz(x+z-x-y) - (y+z)[xz - xy] + 1(xz(x+y) - xy(x+z$

$\Rightarrow yz(z-y) - (y+z)(xz - xy) + x(zx + zy - yx - yz)$

$\Rightarrow yz^2 - zy^2 - \cancel{xyz} + xy^2 - xz^2 + \cancel{zx^2}\ \cancel{xyz} + \cancel{xyz} - yx^2 - \cancel{xyz}$

$\Rightarrow z^2(y-x) + y^2(x-z) + x^2(z-y)$

$\Rightarrow y(z^2-y^2) + x(y^2-z^2) + z(x^2-y^2)$

$\Rightarrow y(z+y)(z-y) + x(y+z)(y-z) + z(x-y)(x+y)$

$\cancel{(y+z)}$

Ques: If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_1 x_3}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$, show that

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \partial y_1/\partial x_3 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \partial y_2/\partial x_3 \\ \partial y_3/\partial x_1 & \partial y_3/\partial x_2 & \partial y_3/\partial x_3 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & x_3/x_1 & x_2/x_1 \\ x_3/x_2 & -x_1 x_3/x^2 & x_1/x_2 \\ x_2/x_3 & x_1/x_3 & -x_1 x_2/x_3^2 \end{vmatrix}$$

$$\Rightarrow \frac{1}{x_1 x_2 x_3} \begin{vmatrix} -\frac{x_2 x_3}{x_1} & x_3 & x_2 \\ x_3 & -\frac{x_1 x_3}{x_2} & x_1 \\ x_2 & x_1 & -\frac{x_1 x_2}{x_3} \end{vmatrix}$$

$$\Rightarrow \frac{1}{x_1 x_2 x_3} \left[-\frac{x_2 x_3}{x_1} \left[x_1^2 - x_1^2 \right] - x_3 \left[-x_1 x_2 - x_1 x_2 \right] + x_2 \left[x_3 x_1 + x_1 x_3 \right] \right]$$

$$\Rightarrow \frac{1}{x_1 x_2 x_3} \times \left[2x_1 x_2 x_3 + 2x_1 x_2 x_3 \right] = 4$$

Ques: Verify the chain rule for Jacobians if $x = u$, $y = u\tan v$, $z = w$

$$J_1 = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} \qquad J_2 = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

$$J_1 \Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ \tan v & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix}, \qquad J_2 = \begin{matrix} 1 & \cot v & 0 \\ 0 & \frac{1}{u\sec^2 v} & 0 \\ 0 & 0 & 1 \end{matrix}$$

$$\Rightarrow u\sec^2 v \cancel{\tan v}$$

$$\Rightarrow J_1 \times J_2 = u\sec^2 v \times \frac{1}{u\sec^2 v} = 1$$

Hence, proved.