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$$f(x,y) = f(0,0) + \frac{1}{1!}\left[x\, f_x(0,0) + y\, f_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

Now, $f_x = e^x \sin y \qquad f_y = e^x \cos y$

$f_{xx} = e^x \sin y, \quad f_{xy} = e^x \cos y, \quad f_{yy} = -e^x \sin y$

$$f(x,y) = 0 + \frac{1}{1!}[y] + \frac{1}{2!}[0 + 2xy + 0]$$

$$= y + xy$$

Q-1 Expand $\tan^{-1}\frac{y}{x}$ in the neighbourhood of $(1,1)$ upto and inclusive of second degree term.

By Taylor's series we have

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$$= f(1,1) + \left[(x-1) f_x(1,1) + (y-1) f_y(1,1)\right] + \frac{1}{2}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1) \cancel{f_{xy}(1,1)}\, f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right]$$

Now, $f(x,y) = \tan^{-1} y/x$

$$f_x = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} = \frac{-1}{2}$$

$$f_y = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{1}{2}$$

$\cancel{fxy}$ $fxx = \dfrac{y(2x)}{(x^2+y^2)^2} = \dfrac{2xy}{(x^2+y^2)^2}$

$fyy = \dfrac{-2xy}{(x^2+y^2)^2}$

$fxy = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$fxx(1,1) = \dfrac{1}{2}$ $\qquad fyy(1,1) = \dfrac{-1}{2}$

$fxy = 0$

Now,

$$f(x,y) = \frac{\pi}{4} + \left[(x-1)\frac{-1}{2} + (y-1)\frac{1}{2}\right] + \frac{1}{2}\left[(x-1)^2\frac{1}{2} + 2(x-1)(y-1)\times 0 + (y-1)^2 \times \frac{-1}{2}\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2$$

Q-3 Find the first six terms of the expansion of $e^x \log(1+y)$ in a Taylor series in the neighbourhood of the point $(0,0)$

Solution:-

By Taylor's series

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)\,fx(a,b) + (y-b)\,fy(a,b)\right]$$
$$+ \frac{1}{2!}\left[(x-a)^2 fxx(a,b) + 2(x-a)(y-b)\,fxy(a,b) + (y-b)^2 fyy(a,b)\right] + \frac{1}{3!}\left[(x-a)^3 fxxx + 3(x-a)^2(y-b)\,fxxy + 3(x-a)(y-b)^2 fxyy + (y-b)^3 f\ldots\right]$$

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$$f(x,y) = f(0,0) + \left[x \cdot f_x(0,0) + y\, f_y(0,0) \right] + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2(xy) \cdot f_{xy}(0,0) + y^2 f_{yy}(0,0) \right]$$

$$+ \frac{1}{3 \times 2}\left[x^3 f_{xxx}(0,0) + 3xy^2 f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0) \right]$$

$f_x = e^x(\log(1+y)) = 0$

$f_y = \dfrac{e^x}{1+y} = 1$

$f_{xx} = e^x(\log(1+y)) = 0$

$f_{yy} = \dfrac{-e^x}{(1+y)^2} = -1$

$f_{xy} = \dfrac{e^x}{1+y} = 1$

$f_{xxx} = e^x \log(1+y) = 0$

$f_{yyy} = \dfrac{e^x(2(1+y))}{(1+y)^4} = 2$

$f_{xxy} = \dfrac{e^x}{1+y} = 1$

$f_{xyy} = \dfrac{-e^x}{(1+y)^2} = -1$

$$f(x,y) = 0 + [0 + y] + \frac{1}{2}[0 + 2xy + y^2(-1)]$$

$$+ \frac{1}{6}[0 + 3x^2y + 3xy^2(-1) + y^3(2)]$$

$$= y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \frac{y^3}{3}$$