16.03.23

Lecture - 24

Maxima and minima for the function of three variables :-

Let $u = f(x, y, z)$.

- Find $\frac{\partial y}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$ and equate them to zero.
- Find $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial z^2}$, $\frac{\partial^2 u}{\partial y \partial z}$, $\frac{\partial^2 u}{\partial z \partial x}$, $\frac{\partial^2 u}{\partial x \partial y}$

they are denoted by A, B, C, F, G, H respectively.

- Also find

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix} = D_1 \text{ (say) and find}$$

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = D_2 \text{ (say)}$$

The given function u will have a minimum if $A > 0$, $D_1 > 0$, $D_2 > 0$. And will have a maximum if, $A < 0$, $D_1 > 0$, $D_2 < 0$

- If these above conditions are not satisfied, we have niether maximum nor minimum.

Question-1 Examine for extreme values

$$f(x, y, z) = x^2 + y^2 + z^2 - xy + x - 2z.$$

$$\frac{\partial u}{\partial x} = 2x - y + 1$$

$$\frac{\partial u}{\partial y} = 2y - x, \quad \frac{\partial u}{\partial z} = 2z - 2.$$

Now, on solving

$$2x - y + 1 = 0 \quad \text{--- 1.}$$

$$2y - x = 0 \quad \text{--- 2.}$$

$$2z - 2 = 0 \quad \text{--- 3.}$$

$$z = 1, \quad y = -1/3, \quad x = -2/3.$$

$$A = \frac{\partial^2 u}{\partial x^2} = 2, \quad B = \frac{\partial^2 u}{\partial y^2} = 2,$$

$$C = \frac{\partial^2 u}{\partial z^2} = 2.$$

$$F = \frac{\partial^2 u}{\partial y \partial z} = 0, \quad G = \frac{\partial^2 u}{\partial z \partial x} = 0$$

$$H = \frac{\partial^2 u}{\partial x \partial y} = -1$$

$$D_1 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4 - 1)$$

$$= 3 > 0$$

$$D_2 = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 6 > 0.$$

$\because$ $A > 0$, $D_1 > 0$ and $D_2 > 0$, therefore the above function will give minimum values at point $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$

Now,

minimum value $= \frac{4}{9} + \frac{1}{9} + 1 - \frac{2}{9} - \frac{2}{3} - 2$

$$= \frac{4 + 1 + 9 - 2 - 6 - 18}{9} = \frac{-12}{9}$$

$$= -4/3. \text{ Ans.}$$

Lagrange Method of Multiplier:-

To find the maximum or minimum values of a function of three or more variables when the variables are not independent but are connected by some given relation we try to convert the given function to the one, having least number of independent variables with the help of given relation.

When this procedure is not practicable, we use Lagrange method.

Let $f(x, y, z)$ be a function of x, y, z which is to be examined for maximum or minimum value.

- Let the variables (x, y, z) be connected by the relation $\phi(x, y, z) = 0$ ---------- 1.
- For $f(x, y, z)$ to have a max or min value, the necessary condition is →

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0$$

- $$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0 \text{ ---------- } 2.$$

Also from 1. taking differential we get,

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0 \text{ ---------- } 3.$$

Multiplying equation 3 by a parameter "λ" and adding to equation 2, we get

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right) dz = 0$$

This equation will hold good.

If $\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

these equations together with equation 1. give the values of x, y, z and λ for a maximum or minimum