20.03.23 Lecture-25

$F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$

For stationary value of F,

$dF = 0.$

$= \left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right)$

$= 0\,dx + 0\,dy + 0\,dz.$

$=$ Now,

$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$

$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0.$

$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0.$

Question-1 Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0.$

Now, $\phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

Let $2x$, $2y$ and $2z$ be the length, breadth and height of the rectangular parallelopiped respectively

$\therefore V = 8xyz.$

Consider the lagrange's function,

$L = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$

For stationary value of L,

$dF = 0.$

$= \left[8yz + \lambda\left(\frac{2x}{a^2}\right)\right] dx + \left[8xz + \lambda\left(\frac{2y}{b^2}\right)\right] dy +$

$\left[8xy + \lambda\left(\frac{2z}{c^2}\right)\right] dz = 0$

$8yz + \frac{2\lambda x}{a^2} = 0$ ... 1.

$8xz + \frac{2\lambda y}{b^2} = 0$ ... 2.

$8xy + \frac{2\lambda z}{c^2} = 0$ ... 3.

On multiplying x in eq. 1, y in 2, z in 3, we get,

$8xyz + 2\lambda x^2 / a^2 = 0$

$8xyz + 2\lambda y^2 / b^2 = 0$

$8xy + 2\lambda z^2 / c^2 = 0.$

On adding above three equation,

$$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0.$$

$$24xyz + 2\lambda = 0.$$

$$-\lambda = 12xyz,$$

$$\lambda = -12xyz.$$

Now, putting value of λ in eq. 1.

$$8yz + \frac{2(-12xyz)x}{a^2} = 0$$

$$8yz\left(1 - \frac{3x^2}{a^2}\right) = 0.$$

$$1 - \frac{3x^2}{a^2} = 0$$

$$\Rightarrow x = a/\sqrt{3}.$$

Similarly, $y = b/\sqrt{3}.$

$z = c/\sqrt{3}.$

Now, $\lambda = -12xyz$

$$= -\frac{4abc}{\sqrt{3}}$$

Now,

Volume of the parallelopied. $= 8xyz$

$$= 8 \cdot \frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \frac{c}{\sqrt{3}}$$

$$= \frac{8abc}{3\sqrt{3}} \text{ unit cube Ans.}$$