21.03.23

Lecture-26

Question-1 Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

Let $2x$, $2y$, $2z$ be the length, breadth and height respectively for the rectangular solid. And let $R$ be the radius of the sphere.

Now,

$$x^2 + y^2 + z^2 = R^2$$

$$\phi(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0$$

$$V = 8xyz$$

For stationary value

$$dF = 0$$

Here, $F = 8xyz + \lambda(x^2 + y^2 + z^2 - R^2)$

$$\Rightarrow [8yz + \lambda(2x)]\,dx + [8xz + \lambda(2y)]\,dy + [8xy + \lambda(2z)]\,dz = 0.$$

From above,

$$8yz + \lambda 2x = 0 \quad \text{---} \quad 1.$$

$$8xz + \lambda 2y = 0 \quad \text{---} \quad 2.$$

$$8xy + \lambda 2z = 0 \quad \text{---} \quad 3.$$

On multiplying $x$ in 1., $y$ in 2., $z$ in 3. and adding them,

$$24xyz + 2\lambda(x^2 + y^2 + z^2) = 0.$$

$$\Rightarrow 24xyz + 2\lambda(R^2) = 0.$$

$$12xy + \lambda R^2 = 0.$$

$$\lambda = -\frac{12xy}{R^2}$$

On putting this value in 1. and solving,

$$x = y = z.$$

Hence, the rectangular is cube. Ans.

<u>Question-2</u> If $x$ and $y$ satisfy the relation $ax^2 + by^2 = ab$, prove that extreme values of the function $u = x^2 + xy + y^2$ are given by the roots of equation

$$4(u-a)(u-b) = ab.$$

Now,

$$ax^2 + by^2 = ab.$$

$$ax^2 + by^2 - ab = 0.$$

Now, $u = x^2 + y^2 + xy$.

Consider the lagrange's function,

$$F = x^2 + y^2 + xy + \lambda\left(\frac{x^2}{b} + \frac{y^2}{a} - 1\right).$$

For stationary value,

$$dF = 0$$

$$= \left[2x + y + \lambda\left(\frac{2x}{b}\right)\right] dx + \left[2y + x + \lambda\left(\frac{2y}{a}\right)\right] dy =$$

on compairing,

$$2x + y + \lambda\left(\frac{2x}{b}\right) = 0.$$

$$2xb + yb + \lambda 2x = 0 \text{ ------ } 1.$$

$$2ya + xa + \lambda 2y = 0 \text{ ----- } 2.$$

on multiplying $x$ in 1. and $y$ in 2. and adding them.

$$= 2x^2 + xy + \frac{2\lambda x^2}{b} = 0$$

$$xy + 2y^2 + \frac{2\lambda y^2}{a} = 0.$$

$$= 2(x^2 + y^2 + xy) + 2\lambda\left(\frac{x^2}{b} + \frac{y^2}{a}\right) = 0.$$

$$= 2u + 2\lambda = 0$$

$$\lambda = -u$$

Now, from 1.

$$2x + y - \frac{2ux}{b} = 0.$$

$$= 2x\left(1 - \frac{u}{b}\right) + y = 0 \text{ ------ } 3.$$

Now, From 2.

$$= x + 2y - \frac{2uy}{a} = 0$$

$$= 2y\left(1 - \frac{u}{a}\right) + x = 0 \quad \ldots\ldots\ldots 4.$$

From 3.,

$$\frac{x}{y} = \frac{-1}{2(1 - u/b)}$$

From 4.,

$$\frac{x}{y} = -2(1 - u/a).$$

From above,

$$\frac{-1}{2(1-u/b)} = -2(1-u/a).$$

$$4\left(1 - \frac{u}{a}\right)\left(1 - \frac{u}{b}\right) = 1.$$

$$4(u-a)(u-b) = ab.$$

Hence Proved. Ans.