24|03|23

Q.4 Evaluate, $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{(1+x^2)+y^2}$

Sol. $\int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$

$\Rightarrow \int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot \frac{\pi}{4} \cdot dx$

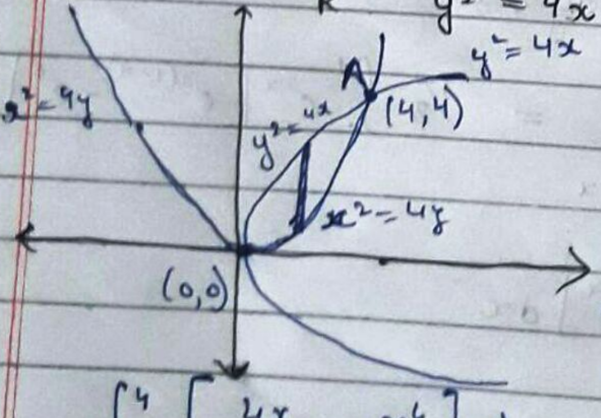
$\Rightarrow \frac{\pi}{4}\left[\log|x+\sqrt{1+x^2}|\right]_0^1$

$\Rightarrow \frac{\pi}{4}\left[\log|1+\sqrt{2}| - \log 1\right]$

$\Rightarrow \frac{\pi}{4}\left[\log|1+\sqrt{2}|\right]$

Q.5 Evaluate $\iint_R y\,dx\,dy$, where R is region bounded by the pa[rabolas] $y^2=4x$ and $x^2=4y$

Sol.



$\Rightarrow \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} y\,dy\,dx$

$\Rightarrow \int_0^4 \left[\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}} dx$

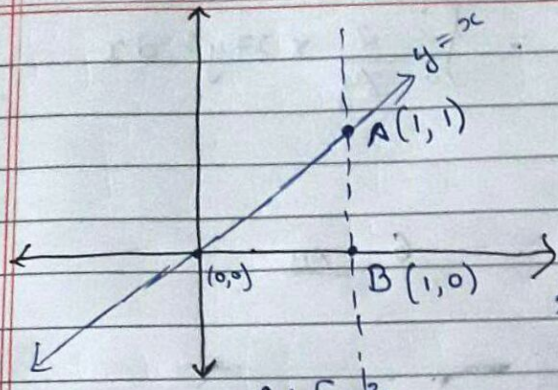
$\Rightarrow \int_0^4 \left[\frac{4x}{2} - \frac{x^4}{32}\right] dx$

$\Rightarrow \left[\frac{4x^2}{4} - \frac{x^5}{160}\right]_0^4 = 16 - \frac{\cancel{2^5} \times 2^5}{\cancel{2^4} \times 10}$

$= 16 - \frac{32}{5} = \frac{48}{5}$

Q.6 When the region R of integration is triangle given by $y=0$, $y=x$ and $x=1$. Show that $\iint_R \sqrt{4x^2-y^2}\,dx\,dy = \frac{1}{3}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$

Sol.



$$\Rightarrow \int_0^1\int_0^x \sqrt{4x^2-y^2}\,dy\,dx$$

$$\Rightarrow \int_0^1\int_0^x \sqrt{(2x)^2-y^2}\,dy\,dx$$

$$\Rightarrow \int_0^1 \left[\frac{y}{2}\sqrt{4x^2-y^2} + \frac{4x^2}{2}\sin^{-1}\frac{y}{2x}\right]_0^x dx$$

$$\Rightarrow \int_0^1 \left[\frac{\sqrt{3}x^2}{2} + 2x^2\sin^{-1}\left(\frac{1}{2}\right) - 2x^2(0) - 0\right]$$

$$\Rightarrow \int_0^1 \left[\frac{\sqrt{3}\,x^2}{2} + \frac{x^2\pi}{3} \cancel{-2x^2}\right] dx$$

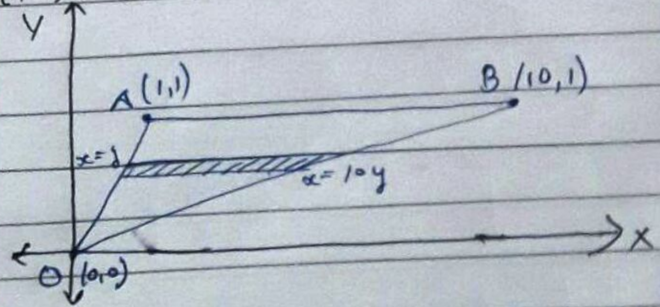
$$\Rightarrow \int_0^1 \left[\frac{\sqrt{3}x^2}{2} + \frac{x^2\pi}{3}\right] dx$$

$$\Rightarrow \left[\frac{\sqrt{3}}{2}\cdot\frac{x^3}{3} + \frac{x^3\pi}{9}\right]_0^1 = \frac{\sqrt{3}}{6} + \frac{\pi}{9}$$

$$= \frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) \quad \text{— Hence Proved.}$$

Q.7 Evaluate $\iint_S \sqrt{xy-y^2}\,dx\,dy$, where S is triangle with vertices (0,0), (10, 1) and (1,1).

Sol.



For line OA,

$$y - y_1 = \frac{(x_2 - x_1)}{x_2 - x_1}(y_2 - y_1)$$

$$y - 0 = \frac{x-0}{1}(1)$$

$$\boxed{y = x}$$

For line OB, $y - 0 = \dfrac{x-0}{10}(1) \Rightarrow \boxed{x = 10y}$

$$\Rightarrow \int_0^1\int_y^{10y} \sqrt{xy-y^2}\,dx\,dy = \int_0^1 \left[\frac{2}{3y}(xy-y^2)^{3/2}\right]_y^{10y} dy$$

$\rightarrow \int_0^1 \frac{2}{3y}\left[9y^2\right]^{3/2} dx = \int_0^1 \frac{2}{\cancel{3}y} \times \cancel{27}^{9} y^3 \, dx$

$= \int_0^1 18y^2 \, dx$

$= 18\left[\frac{y^3}{3}\right]_0^1 = 6$ Ans