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Lecture-6

Question-1 Verify Euler's Theorem for the function $u = \sin^{-1} x/y + \tan^{-1} y/x$.

$$u = \sin^{-1} x/y + \tan^{-1} y/x$$
$$= \sin^{-1} x/y + \tan^{-1} 1/x/y.$$
$$= y^0 \phi(x/y).$$

This implies that u is a homogeneous function of degree 0 in variables x and y.

$\therefore$ By Euler's theorem, we have to show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \times u = 0.$$

Now, differentiating u with respect to x,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-x^2/y^2}} \cdot \frac{1}{y} + \frac{1}{1+y^2/x^2} \cdot \left(\frac{-y}{x^2}\right).$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}.$$

Now,

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{---------- } 1.$$

Now, differentiating u with respect to y,

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-x^2/y^2}}\left(\frac{-x}{y^2}\right) + \frac{1}{1+y^2/x^2} \cdot \frac{1}{x}.$$

$$\frac{\partial u}{\partial y} = \frac{-x \cdot y}{y^2\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}.$$

Now,

$$y\frac{\partial u}{\partial y} = \frac{-xy}{y\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}.$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{\cancel{\sqrt{x^2+y^2}}} \quad \text{-------- } 2.$$

On adding 1. and 2.,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

$$0 = 0$$

LHS = RHS

Hence proved. Ans.

Question-2 If $u = (x^{1/4} + y^{1/4})(x^{1/5} + y^{1/5})$, apply Euler's theorem to find the value of

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}.$$

$$u = x^{1/4}\left(1+\left(\frac{y}{x}\right)^{1/4}\right) x^{1/5}\left(1+\left(\frac{y}{x}\right)^{1/5}\right)$$

$$= x^{1/4+1/5}\,\phi(y/x)$$

$$= x^{9/20}\,\phi(y/x).$$

This implies that u is a homogeneous function of degree $\frac{9}{20}$ in variable x and y.

Now, we need to prove by Euler's theorem:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{9}{20}u.$$

Now,

$$\frac{\partial u}{\partial x} = \frac{1}{5}(x^{1/4} + y^{1/4})x^{-4/5} + \frac{1}{4}(x^{1/5} + y^{1/5})x^{-3/4}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{5}(x^{1/4} + y^{1/4})x^{-4/5} + \frac{x}{4}(x^{1/5} + y^{1/5})x^{-3/4}$$

$$= \frac{x^{1/5}}{5}(x^{1/4} + y^{1/4}) + \frac{x^{1/4}}{4}(x^{1/5} + y^{1/5}) \quad \text{------- } 1.$$

Now,

$$y\frac{\partial u}{\partial y} = \frac{y^{1/5}}{5}(x^{1/4} + y^{1/4}) + \frac{y^{1/4}}{4}(x^{1/5} + y^{1/5}) \quad \text{------- } 2.$$

Adding eq. 1 and 2.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= \frac{9}{20}(x^{1/4} + y^{1/4})(x^{1/5} + y^{1/5})$$

Hence Proved. Ans.