Question-1 If $u = \cos\left(\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

$$u(tx, ty, tz) = \cos\left(\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$$

$$= u(x, y, z).$$

$$= t^0 g(x, y, z)$$

This implies that u is a homogeneous function of degree 0 in variable x, y, z

Hence, by Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = 0 \times u$$

$$= 0$$

Hence proved. Ans.

Question-2 Show that:-

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\tan u, \text{ where}$$

$$u = \sin^{-1}\left(\frac{x^3 + y^3 + z^3}{ax + by + cz}\right).$$

$$u(tx, ty, tz) = \sin^{-1}\left(\frac{(t^2)(x^3 + y^3 + z^3)}{ax + by + cz}\right)$$

$$= \cancel{t^2} \sin^{-1}\left(\frac{x^3 + y^3 + z^3(t^2)}{ax + by + cz}\right).$$

$$\neq t^n u(x, y, z).$$

This implies that the given function is not ~~fun~~ homogeneous.

Now,

$$\sin u = \frac{x^3 + y^3 + z^3}{ax + by + cz}$$

$$\Rightarrow \sin u = \frac{x^3 \left(1 + (y/x)^3 + (z/x)^3\right)}{x \left(a + by/x + cz/x\right)}.$$

$$\sin u = x^2 \phi\left(\frac{y}{x}, \frac{z}{x}\right)$$

This implies that the function,

$$\sin u = \frac{x^3 + y^3 + z^3}{ax + by + cz}.$$

is a homogeneous function of degree 2 in variables x, y, z.

Hence, By Euler's Theorem

$$= x \frac{\partial}{\partial x}(\sin u) + y \frac{\partial}{\partial y}(\sin u) + z \frac{\partial (\sin u)}{\partial z} = 2 \sin u$$

$$= x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} + z \cos u \frac{\partial u}{\partial z} = 2 \sin u$$

$$= \cos u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}\right) = 2 \sin u$$

$$= x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 2 \tan u.$$

Hence Proved. Ans.

Question-3 If $u = \tan^{-1} \frac{x^3 + y^3}{x - y}$, prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$$

Now, the given function u is not a homogeneous function. But

$$\tan u = \frac{x^3 + y^3}{x - y}.$$

$$= \frac{x^3(1 + (y/x)^3)}{x(1 - y/x)}.$$

$$= x^2 \phi(y/x).$$

This implies that now, the function is homogeneous of degree 2 in variable x, y

Hence, By Euler's theorem,

$$= x \frac{\partial (\tan u)}{\partial x} + y \frac{\partial}{\partial y}(\tan u) = 2 \tan u.$$

$$= \sec^2 u \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right) = 2\tan u.$$

$$= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u \cos^2 u$$

$$= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u \cos u.$$

$$= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$$

Hence Proved. Ans.

Question-4 If $u = \tan^{-1} \frac{x^3+y^3}{x-y}$, prove that.

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \sin 4u - \sin 2u.$$

Now, the given function u is not a homogeneous function, But

$$\tan u = \frac{x^3+y^3}{x-y}.$$

$$= \frac{x^3 (1 + (y/x)^3)}{x (1 - y/x)}$$

$$= x^2 \phi (y/x).$$

This implies that the function is homogeneous of degree 2 in variable x, y.

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \phi(u) [\phi'(u) - 1]$$

$$= \sin 2u [2\cos 2u - 1]$$

$$= 2\sin 2u \cos 2u - \sin 2u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \sin 4u - \sin 2u.$$

Hence Proved. Ans.

Question-5 If $u = \log \left(\frac{x^4+y^4}{x+y} \right)$, Show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3.$$

Now, the given function u is not a homogeneous