Question-1 If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that $x\frac{du}{dx} + y\frac{du}{dy} = \frac{1}{2}\tan u$.

Now, the above function is not homogeneous
But,

$$\sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$
$$= x^{1/2}\phi(y/x)$$

Hence the above function is a homogeneous function of degree 1/2 in variable x/y.
By Euler's Theorem,

$$\cos u\left(x\frac{du}{dx} + y\frac{du}{dy}\right) = \frac{1}{2}\sin u$$

$$= x\frac{du}{dx} + y\frac{du}{dy} = \frac{1}{2}\tan u.$$

Hence Proved. Ans.

Question-2. If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u\cos u}{4\cos^3 u}$$

Now, By Euler's Theorem,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} \cdot \frac{1}{2}\tan u\left[\frac{\sec^2 u}{2} - 1\right].$$

$$= \frac{\tan u}{2}\left(\frac{\sec^2 u - 2}{2}\right)$$

$$= \frac{\sin u}{\cos u}\left(\frac{1 - 2\cos^2 u}{\cos^2 u}\right)\cdot\frac{1}{4}$$

$$x^2\frac{d^2 u}{dx^2} + 2xy\frac{d^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u\cos 2u}{4\cos^3 u}.$$

Hence Proved. Ans.

Question-3 If $u = x\phi(y/x) + \psi(y/x)$, show that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\phi(y/x)$.

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.

(i) Let $v = x\phi(y/x)$ and $w = \psi(y/x)$.

Hence,

$u = v + w$.

Now, $v$ is a homogeneous function of deg 1 in variable $x$ and $y$.

and $w$ is a homogeneous function of deg 0 in variable $x$ and $y$.

Now, By Euler's Theorem.

$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v$. ------1.

$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0$ ------2.

on adding 1 and 2.

$x\frac{\partial}{\partial x}(v+w) + y\frac{\partial}{\partial y}(v+w) = v$.

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = x\phi(y/x)$. Hence Proved.

(ii) Now,

By Euler's theorem,

$x^2\frac{\partial^2 v}{\partial x^2} + y^2\frac{\partial^2 v}{\partial y^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} = 1(1-1)v = 0$.

$x^2\frac{\partial^2 w}{\partial x^2} + y^2\frac{\partial^2 w}{\partial y^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} = 0(0-1)w = 0$.

------ 2.

on adding 1 and 2,

$x^2\frac{\partial^2}{\partial x^2}(w+v) + y^2\frac{\partial^2}{\partial y^2}(w+v) + 2xy\frac{\partial^2}{\partial x \partial y}(w+v) = 0$

$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = 0$. Hence Proved.

question-4 : Verify Euler's Theorem for the function
$u = x^2 \log(y/x)$
Applying alternate test,
$u(tx, ty) = t^2 x^2 \log\left(\frac{ty}{tx}\right)$.

$= t^2 x^2 \log(y/x)$.

This implies that above function is a homogeneous function of degree 2 in variables $x, y$.

Now, we need to differentiate $u$ with respect to $x$, we get,

$\frac{\partial u}{\partial x} = x^2 \frac{1}{y/x}\left(\frac{-y}{x^2}\right) + \log y/x \cdot 2x$.

$\Rightarrow x\frac{\partial u}{\partial x} = -x^2 + 2x^2 \log\frac{y}{x}$ ------------ 1.

Now, differentiating with respect to $y$ we get,

$\frac{\partial u}{\partial y} = x^2 \cdot \frac{1}{y/x} \cdot \frac{1}{x}$

$y\frac{\partial u}{\partial y} = x^2$ ------------ 2.

On adding 1 and 2,

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -x^2 + 2x^2 \log\frac{y}{x} + x^2$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$

Hence verified. Ans.