Matrix Method to solve system of linear equations

Let us consider the normal form of linear system of n first order diff. eqns in n unknowns functions $x_1, x_2 - - - x_n$.

$$\frac{dx_1}{dt} = a_{11}(t)\, x_1 + a_{12}(t)\, x_2 + - - + a_{1n}(t)\, x_n + F_1(t)$$

$$\frac{dx_2}{dt} = a_{21}(t)\, x_1 + a_{22}(t)\, x_2 + - - + a_{2n}(t)\, x_n + F_2(t)$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}(t)\, x_1 + a_{n2}(t)\, x_2 + \cdots + a_{nn}(t)\, x_n + F_n(t) \qquad \text{— } (1.1)$$

we shall assume that all functions $a_{ij}(t)$ and $F_i(t)$ $i = 1, 2 - - n$, $j = 1, 2 - - n$, are continuous on a real interval $a \le t \le b$.

The above system is called nonhomogeneous linear system and if all $F_i(t) = 0$ $i = 1, 2 - - n$, for all t then this system is called homogeneous.

The linear system (1.1) can be written more compactly as

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}\, x_j + F_i(t) \qquad \text{— } (1.2)$$

$$(i = 1, 2, - - n)$$

further it can be written in an even more compact manner using vectors and matrices as

$$\frac{dx}{dt} = A(t)\, x + F(t) \qquad \text{— } (1.3)$$

where

$$\frac{dx}{dt} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix}$$

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & & \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

$$F(t) = \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

The eq^n (1.3) is knowne as linear vector equation corresponding to the system (1.1). Thus the system (1.1) and the vector diff. eq^n. (1.3) both express the same relations and so are equivalent to one another. Sometimes system (1.1) are known as the scalar form of vector differential eq^n. (1.3).

First we will discuss matrix method to solve a homogeneous linear system

Consider the homogeneous linear system

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned}\right\} \quad (1.4)$$

The corresponding vector diff. eqn. is

$$\frac{dx}{dt} = Ax \quad \text{—— (1.5)}$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

and all a_{ij} are real constants.

Case-I <u>when all eigen values are distinct</u>

Suppose $\lambda_1, \lambda_2 \ldots \lambda_n$ are distinct n eigen values of A and let $\alpha^{(1)}, \alpha^{(2)} \ldots \alpha^{(n)}$ be set of respective corresponding eigenvectors of A. Then on every real interval, the n-vector functions defined by

$$\alpha^{(1)} e^{\lambda_1 t}, \; \alpha^{(2)} e^{\lambda_2 t} \ldots \alpha^{(n)} e^{\lambda_n t}.$$

form a L.I set of solutions of (1.4) i.e. (1.5) and then the general soln of (1.4) is

$$x = c_1 \alpha^{(1)} e^{\lambda_1 t} + c_2 \alpha^{(2)} e^{\lambda_2 t} + \cdots + c_n \alpha^{(n)} e^{\lambda_n t} \quad \text{—— (1.6)}$$

Examples.

Ex1 - Using the matrix method, find the general soln of the system

$$\left.\begin{aligned} \frac{dx_1}{dt} &= 5x_1 - 2x_2 \\ \frac{dx_2}{dt} &= 4x_1 - x_2 \end{aligned}\right\} \text{—— (1)}$$

The corresponding vector diff. eqn is

$$\frac{dx}{dt} = Ax \quad \text{—— (2)}$$

where

$A = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

The characteristic eqn of matrix A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0 \text{ gives } \lambda = 1, 3$$

$\therefore$ The distinct eigen values of A are

$$\lambda_1 = 1, \quad \lambda_2 = 3$$

For eigen vectors, we have $(A - \lambda I)x = 0$

i.e. $\begin{bmatrix} 5-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ — 3

corresponding $\lambda = 1$, from 3

$$\begin{bmatrix} 4 & -2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

it is equivalent to eqn $2x_1 - x_2 = 0$

so $\alpha^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

and corresponding $\lambda = 3$, from 3, we get

$$\begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow x_1 - x_2 = 0$$

so $\alpha^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now the general soln of 2 is

$$x = c_1 \alpha^{(1)} e^{\lambda_1 t} + c_2 \alpha^{(2)} e^{\lambda_2 t}$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

i.e. $x = c_1 \begin{pmatrix} e^{t} \\ 2e^{t} \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$ — 4

where c_1 & c_2 are arbitrary constants.

and the general solution of the given (5)
homo. linear system (1) is

$$x_1 = c_1 e^{t} + c_2 e^{2t}, \quad x_2 = 2c_1 e^{t} + c_2 e^{2t}$$

Ans.

Ex 2 Using vector-matrix method, solve

$$\left.\begin{aligned} \frac{dx_1}{dt} &= 7x_1 - x_2 + 6x_3 \\ \frac{dx_2}{dt} &= -10x_1 + 4x_2 - 12x_3 \\ \frac{dx_3}{dt} &= -2x_1 + x_2 - x_3 \end{aligned}\right\} \quad \text{--- (1)}$$

Soln. The corresponding vector diff. eqn of (1) is

$$\frac{dx}{dt} = Ax \quad \text{--- (2)}$$

where $A = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The characteristic eqn. of A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{vmatrix} = 0$

by expanding, we get $\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$

$$\Rightarrow (\lambda - 2)(\lambda - 3)(\lambda - 5) = 0$$

$$\Rightarrow \lambda = 2, 3, 5$$

$\therefore$ The three distinct eigen values of A are $\lambda_1 = 2$, $\lambda_2 = 3$ and $\lambda_3 = 5$

For corresponding eigenvectors, we have

$$(A - \lambda I)x = 0$$

i.e. $\begin{bmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ --- (3)

when $\lambda=2$, from ③ we get

$$\begin{bmatrix} 5 & -1 & 6 \\ -10 & 2 & -12 \\ -2 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 5x_1-x_2+6x_3=0 \\ 5x_1-x_2+6x_3=0 \\ 2x_1-x_2+3x_3=0 \end{matrix}$$

solving last two we have $\frac{x_1}{-3+6}=\frac{x_2}{12-15}=\frac{x_3}{-5+2}$

i.e. $\frac{x_1}{1}=\frac{x_2}{-1}=\frac{x_3}{-1}$

$\therefore\ \alpha^{(1)}=\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ = eigen vector corresponding $\lambda_1=2$

when $\lambda=3$ $\quad \begin{bmatrix} 4 & -1 & 6 \\ -10 & 1 & -12 \\ -2 & 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{matrix} 4x_1-x_2+6x_3=0 \\ -10x_1+x_2-12x_3=0 \\ -2x_1+x_2-4x_3=0 \end{matrix}$$

Solving first and third we get

$x_1=1,\ x_2=-2,\ x_3=-1$

$\therefore\ \alpha^{(2)}=\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$, eigen vector corresponding $\lambda_2=3$

when $\lambda=5$ from ③, we obtain

$$\begin{bmatrix} 2 & -1 & 6 \\ -10 & -1 & -12 \\ -2 & 1 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 2x_1-x_2+6x_3=0 \\ 10x_1+x_2+12x_3=0 \\ 2x_1-x_2+6x_3=0 \end{matrix}$$

solving last two different eqns we get

$x_1=3 \quad x_2=-6 \quad x_3=-2$

$\therefore\ \alpha^{(3)}=\begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix}$ = eigen vector corresponding $\lambda_3=5$

Since the eigen values of A are distinct therefore L.I. solns of ② are

$\alpha^{(1)}e^{\lambda_1 t},\ \alpha^{(2)}e^{\lambda_2 t},\ \&\ \alpha^{3}e^{\lambda_3 t}$

and the general soln of (2) is

$$x = c_1 \alpha^{(1)} e^{\lambda_1 t} + c_2 \alpha^{(2)} e^{\lambda_2 t} + c_3 \alpha^{(3)} e^{\lambda_3 t}$$

i.e. $$x = c_1 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix} e^{5t}$$

or $$x = c_1 \begin{pmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix} + c_3 \begin{pmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{pmatrix}$$

and the general soln. of the given system is

$$x_1 = c_1 e^{2t} + c_2 e^{3t} + 3c_3 e^{5t}$$

$$x_2 = -c_1 e^{2t} - 2c_2 e^{3t} - 6c_3 e^{5t}$$

$$x_3 = -c_1 e^{2t} - c_2 e^{3t} - 2c_3 e^{5t}$$

where c_1, c_2, c_3 are arbitrary constants.

In next lecture we will discuss the case when A (coeff. matrix) has repeated eigen values.