Case-II when characteristic (eigen) values are repeated:

Suppose λ_1 is repeated m times i.e. λ_1 is an eigen value of multiplicity m and all the other are distinct. further suppose that the repeated eigen value λ_1 of multiplicity m has p L.I. eigen vectors, where $1 \le p \le m$

Now consider two subcases.

(i) $p = m$ (ii) $p < m$

In subcase(i) $p = m$, there are m L.I. eigen vectors $\alpha^{(1)}, \alpha^{(2)} \ldots \alpha^{(m)}$ corresponding to the eigen value λ_1 of multiplicity m then the n functions

$$\alpha^{(1)} e^{\lambda_1 t}, \alpha^{(2)} e^{\lambda_1 t}, \ldots \alpha^{(m)} e^{\lambda_1 t}, \alpha^{(m+1)} e^{\lambda_{m+1} t}, \ldots \alpha^{(n)} e^{\lambda_n t}$$

form a L.I. set of n solutions of the vector differential equation and the general soln. is the linear combination of these n solutions.

(See Ex. 3)

Subcase (ii) - Let λ is an eigen value of multiplicity $m = 2$ and $p = 1$ then there is only one eigen vector α (say) i.e. there is only one solution $\alpha e^{\lambda t}$ corresponding λ. Then the second L.I. soln is of the form

$$(\alpha t + \beta) e^{\lambda t}$$

where β is ~~the~~ a vector which satisfies the equation

$$(A - \lambda I)\beta = \alpha \quad —$$

Therefore in this situation the two L.I solutions corresponding λ are

$$\alpha e^{\lambda t} \text{ and } (\alpha t + \beta) e^{\lambda t}.$$

Now let λ be an eigen value of multiplicity $m = 3$ and $p < m$ so there are two possibilities $P = 1$ and $P = 2$.

<u>If $P = 1$</u> then there is only one eigen value α and hence only one soln. of the form $\alpha e^{\lambda t}$ corresponding λ. Then the second soln. corresponding to λ is of the form

$$(\alpha t + \beta) e^{\lambda t}$$

where α is an eigen ~~value~~ vector corresponding to λ that is satisfied

$$(A - \lambda I)\alpha = 0$$

and β is a vector which satisfies the eqn.

$$(A - \lambda I)\beta = \alpha$$

and the third solution corresponding to λ is the form

$$\left(\alpha \frac{t^2}{2!} + \beta t + \gamma\right) e^{\lambda t}$$

where α and β are as above and γ satisfies

$$(A - \lambda I)\gamma = \beta$$

<u>If $P = 2$</u> then there is two L.D. eigen vectors $\alpha^{(1)}$ and $\alpha^{(2)}$ corresponding to λ and hence there is two L.I. solns. of the form

$$\alpha^{(1)} e^{\lambda t} \text{ and } \alpha^{(2)} e^{\lambda t}$$

then a third soln. corresponding to λ is of the form

$$(\alpha t + \beta) e^{\lambda t}$$

where α satisfies $(A - \lambda I)\alpha = 0$ — (1)

and β satisfies $(A - \lambda I)\beta = \alpha$ — (2)

Here $\alpha = \alpha^{(1)}$ or $\alpha = \alpha^{(2)}$ but for more general soln. we take

$$\alpha = k_1 \alpha^{(1)} + k_2 \alpha^{(2)} \quad \text{--- (3)}$$

where k_1 & k_2 are suitable constants whose values can be determined by substituting (3) in (1) for α. After getting values of k_1 & k_2 find required α and then from (2) we can find value of β.

See Ex. (4)

Ex. 3: Using Matrix method, solve

$$\left.\begin{aligned} \frac{dx_1}{dt} &= 3x_1 + x_2 - x_3 \\ \frac{dx_2}{dt} &= x_1 + 3x_2 - x_3 \\ \frac{dx_3}{dt} &= 3x_1 + 3x_2 - x_3 \end{aligned}\right\} \quad \text{--- (1)}$$

Soln: The corresponding vector differential eqn

$$\frac{dx}{dt} = Ax \quad \text{--- (2)}$$

where $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The characteristic eqn of A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 3 & 3 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$\Rightarrow \lambda = 1, 2, 2.$

$\therefore$ The eigen values are $\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 2$

Here $\lambda = 1$ distinct and $\lambda = 2$ is repeated twice i.e. $\lambda = 2$ is of multiplicity 2.

For eigen vectors we have $\begin{bmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 3 & 3 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

when $\lambda = 1$ $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 3 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{aligned} 2x_1 + x_2 - x_3 &= 0 \\ x_1 + 2x_2 - x_3 &= 0 \\ 3x_1 + 3x_2 - 2x_3 &= 0 \end{aligned}$$

by solving first two we get $x_1 = 1,\ x_2 = 1\ x_3 = 3$

$\therefore$ eigen-vector $\alpha^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ corresponding $\lambda_1 = 1$

When $\lambda = 2$, we have $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

It is equivalent to a single eqn

$$x_1 + x_2 - x_3 = 0$$

then it is obvious that

$$x_1 = 1, \; x_2 = -1, \; x_3 = 0$$

&

$$x_1 = 1 \quad x_2 = 0 \quad x_3 = 1$$

are two distinct (L.I.) solns so that

$$\alpha^{(2)} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ and } \alpha^{(3)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

are distinct (L.I.) eigen ~~values~~ vectors corresponding $\lambda = 2$

It is noted that here $m = 2$ & $p = 2$ i.e. it is a subcase (i)

Therefore corresponding repeated eigen value $\lambda = 2$ there are two L.I. solutions

$$\alpha^{(2)} e^{2t} \text{ and } \alpha^{(3)} e^{2t}$$

Thus the three solns are

$$\alpha^{(1)} e^{t}, \; \alpha^{(2)} e^{2t} \;\&\; \alpha^{(3)} e^{2t}$$

i.e. $\begin{pmatrix} e^t \\ e^t \\ 3e^t \end{pmatrix}, \begin{pmatrix} e^{2t} \\ -e^{2t} \\ 0 \end{pmatrix}, \begin{pmatrix} e^{2t} \\ 0 \\ e^{2t} \end{pmatrix}$

and the general soln of (2) is

$$x = c_1 \begin{pmatrix} e^t \\ e^t \\ 3e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{2t} \\ -e^{2t} \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} e^{2t} \\ 0 \\ e^{2t} \end{pmatrix}$$

Finally the general soln of the given system is

$$x_1 = c_1 e^t + c_2 e^{2t} + c_3 e^{2t}$$
$$x_2 = c_1 e^t - c_2 e^{2t}$$
$$x_3 = 3c_1 e^t + c_3 e^{2t}$$

Ans.

Ex. 4 - Using matrix Method, solve

$$\left.\begin{aligned}\frac{dx_1}{dt} &= 4x_1+3x_2+x_3\\ \frac{dx_2}{dt} &= -4x_1-4x_2-2x_3\\ \frac{dx_3}{dt} &= 8x_1+12x_2+6x_3\end{aligned}\right\} \quad -(1)$$

It matrix form

$$\frac{dx}{dt} = Ax \quad -(2)$$

where $A = \begin{bmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The characteristic eqn. of A is $|A-\lambda I|=0$

i.e. $\begin{vmatrix} 4-\lambda & 3 & 1 \\ -4 & -4-\lambda & -2 \\ 8 & 12 & 6-\lambda \end{vmatrix} = 0$

on expanding we get $\lambda^3-6\lambda^2+12\lambda-8=0$

$\Rightarrow \lambda = 2, 2, 2$

That is $\lambda=2$ is an eigenvalue of multiplicity 3

To find eigenvector(s) corresponding $\lambda=2$, we have

$$\begin{bmatrix} 2 & 3 & 1 \\ -4 & -6 & -2 \\ 8 & 12 & 4 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is equivalent to a single eqn.

$$2\alpha_1+3\alpha_2+\alpha_3 = 0 \quad -(3)$$

obviously $\quad \alpha_1 = 1 \quad \alpha_2 = 0 \quad \alpha_3 = -2$

and $\quad \alpha_1 = 0 \quad \alpha_2 = 1 \quad \alpha_3 = -3$

are two distinct solutions of (3) i.e. the two distinct eigen vectors corresponding $\lambda = 2$ are

$$\alpha^{(1)} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \ \& \ \alpha^{(2)} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

Furthermore it is easy to see that these two eigen vectors $\alpha^{(1)}$ and $\alpha^{(2)}$ are L.I., whereas every set of three eigen vectors corresponding to $\lambda = 2$ are L.D. Thus the eigen value $\lambda = 2$ of multiplicity $m = 3$ has $p = 2$ L.I. eigen vectors $\alpha^{(1)}$ and $\alpha^{(2)}$ so that there are two L.I. solutions of (2) are

$$\alpha^{(1)} e^{2t} \quad \text{and} \quad \alpha^{(2)} e^{2t}$$

i.e. $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} e^{2t}$ and $\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} e^{2t}$

or $\begin{bmatrix} e^{2t} \\ 0 \\ -2e^{2t} \end{bmatrix}$ and $\begin{bmatrix} 0 \\ e^{2t} \\ -3e^{2t} \end{bmatrix}$ — (4)

A third solution corresponding $\lambda = 2$ is of the form

$$(\alpha t + \beta) e^{2t} \quad \text{— (5)}$$

where α satisfies $(A - 2I)\alpha = 0$ — (6)

and β satisfies $(A - 2I)\beta = \alpha$ — (7)

Since $\alpha^{(1)}$ and $\alpha^{(2)}$ both are eigenvectors corresponding $\lambda = 2$ so they both satisfy (6) but we need to use the more general

Solution

$$\alpha = k_1\alpha^{(1)} + k_2\alpha^{(2)} \quad — (8)$$

of (6) in order to obtain a nontrivial solⁿ for β in (7)

Thus we have

$$\alpha = k_1\alpha^{(1)} + k_2\alpha^{(2)} = \begin{pmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{pmatrix} \; \& \; \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

and then (7) becomes

$$\begin{bmatrix} 2 & 3 & 1 \\ -4 & -6 & -2 \\ 8 & 12 & 4 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{bmatrix}$$

$$\Rightarrow \quad 2\beta_1 + 3\beta_2 + \beta_3 = k_1$$
$$-4\beta_1 - 6\beta_2 - 2\beta_3 = k_2$$
$$8\beta_1 + 12\beta_2 + 4\beta_3 = -2k_1 - 3k_2$$

from these we get $k_2 = -2k_1 \Rightarrow k_1 = 1 \quad k_2 = -2$

as a simple non trivial solⁿ.

using it in (8), we get $\alpha = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

Now relations (7) becomes

$$2\beta_1 + 3\beta_2 + \beta_3 = 1$$
$$-4\beta_1 - 6\beta_2 - 2\beta_3 = -2$$
$$8\beta_1 + 12\beta_2 + 4\beta_3 = 4$$

Each of these equivalent to

$$2\beta_1 + 3\beta_2 + \beta_3 = 1$$

& a non trivial solⁿ is $\beta_1 = 0, \beta_2 = 0$

$\beta_3 = 1 \qquad \therefore \beta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

putting these values of α and β in (5), the third solⁿ is

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$$\left(\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) e^{2t}$$

i.e. $\begin{bmatrix} te^{2t} \\ -2te^{2t} \\ (4t+1)e^{2t} \end{bmatrix}$ — (9)

Now the three solutions given by (4) & (9) are L.I and the general soln of (2) is

$$x = c_1 \begin{bmatrix} e^{2t} \\ 0 \\ -2e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{2t} \\ -3e^{2t} \end{bmatrix} + c_3 \begin{bmatrix} te^{2t} \\ -2te^{2t} \\ (4t+1)e^{2t} \end{bmatrix}$$

Hence the general soln. of the given system (1) is

$$x_1 = c_1 e^{2t} + c_3 t e^{2t}$$

$$x_2 = c_2 e^{2t} - 2c_3 t e^{2t}$$

$$x_3 = -2c_1 e^{2t} - 3c_2 e^{2t} + c_3 (4t+1) e^{2t}$$

where c_1, c_2, c_3 are arbitrary constants.