## System of linear Ordinary Differential Equations

Consider $\frac{dx}{dt} = ax + by$

$\frac{dy}{dt} = cx + dy$

$X = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad \frac{d}{dt}[X] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} X$

$$\boxed{\frac{dX}{dt} = AX} \quad (*) \qquad X = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

**Working Rule to find the general solution of (*)**

**Step 1**

characteristic eq$^n$ $|A - \lambda I| = 0$

ie $2\times2$, $\lambda^2 - (a+d)\lambda + (ad-bc) = 0$

find roots $\lambda = \lambda_1, \lambda_2$

**Case 1** If $\lambda_1, \lambda_2$ are real and distinct find eigen vectors say $v_1, v_2$ s.t.

$$(A - \lambda_1 I)v_1 = 0 \text{ and } (A - \lambda_2 I)v_2 = 0$$

$\therefore$ sol$^n$ of (*) is

$$\boxed{X(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}}$$

Case② If $\lambda_1 = \lambda_2$ (repeated root) and dimension of eigenspace of $\lambda_1=\lambda_2=\lambda$ is one
then
suppose one eigen vector is $v_1$
now
construct a new vector $v_2$ which satisfy

$$(A-\lambda I)\,v_2 = v_1 .$$

Then two L.I. solutions of (*) are given by
$v_1 e^{\lambda t}$ and $(t v_1 + v_2)e^{\lambda t}$ so

general solution is

$$\boxed{x(t) = c_1 v_1 e^{\lambda t} + c_2 (t v_1 + v_2) e^{\lambda t}}$$

Case③ If $\lambda_1 = \lambda_2$ and dimension of eigenspace of $\lambda_1=\lambda_2=\lambda$ is two
so general solution is

$$\boxed{x(t) = c_1 v_1 e^{\lambda t} + c_2 v_2 e^{\lambda t}}$$

Case④ If $\lambda_1$ and $\lambda_2$ are complex and distinct roots

find one $sol^n$ $v_1 e^{\lambda_1 t}$ s.t. $(A-\lambda_1)v_1 = 0$

if $v_1 e^{\lambda t} = \omega_1 + i\omega_2$

then general solution is

$$\boxed{x(t) = c_1 \omega_1 + c_2 \omega_2}$$

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# Algorithm to solve Linear Systems with constant coefficients :-

Given System is

$$\frac{dx}{dt} = a_1 x + b_1 y$$
$$\frac{dy}{dt} = a_2 x + b_2 y$$

$a_1, a_2, b_1, b_2$ are given constants

(1) Matrix form :

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\boxed{\frac{dX}{dt} = AX} \quad (*)$$

(2) Characteristic eq$^n$ (auxilliary eq$^n$) to find sol$^n$ of (*) is

$x = Ae^{mt}$
$y = Be^{mt}$
$\frac{dx}{dt} = Ame^{mt}$
$\frac{dy}{dt} = Bme^{mt}$
$Ame^{mt} = a_1 Ae^{mt} + b_1 Be^{mt}$

$$\begin{vmatrix} a_1 - m & b_1 \\ a_2 & b_2 - m \end{vmatrix} = 0$$

$$\Rightarrow m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0$$

$A(m - a_1)$
$(a_1 - m)A + b_1 B = 0$
$a_2 A + (b_2 - m)B = 0$

roots of auxilliary eq$^n$ decides the sol$^n$ of (*)

$\begin{vmatrix} a_1 - m & b_1 \\ a_2 & b_2 - m \end{vmatrix} = 0$

(i) If roots are real and distinct say $m_1 \neq m_2$

then sol$^n$ of (*) is

$$x(t) = c_1 \alpha_1 e^{m_1 t} + c_2 \alpha_2 e^{m_2 t}$$

wher $(A - m_1 I)\alpha_1 = 0$

Example:-

solve $\frac{dx}{dt} = x+y$

$\frac{dy}{dt} = 4x-2y$

$A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$

$\therefore$ auxilliary eqⁿ is $\begin{vmatrix} 1-m & 1 \\ 4 & -2-m \end{vmatrix} = 0$

ie $m^2 - (-2+1)m + (-2-4) = 0$

$m^2 + m - 6 = 0$

$m^2 + 3m - 2m - 6 = 0$

$m(m+3) - 2(m+3) = 0$

$(m-2)(m+3) = 0$

$\boxed{m = 2, -3}$ real and distinct roots

$\therefore$ solⁿ is ~~$x(t) = c_1 \alpha_1 e^{2t} + c_2$~~

$$\boxed{x(t) = c_1 \alpha_1 e^{m_1 t} + c_2 \alpha_2 e^{m_2 t}}$$

$\alpha_1 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

where

$(A - m_1)\alpha_1 = 0$ $\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$m_1 = 2$

$\left.\begin{array}{l} -a_1 + a_2 = 0 \\ 4a_1 - 4a_2 = 0 \end{array}\right]$ $a_2 = a_1 = 1$

$2t$

$\therefore$ $\alpha_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(A - m_2)(\alpha_2) = 0$ $\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$m_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ $m_2 = -3$

$4b_1 + b_2 = 0$

$\therefore b_1 = 1, \ b_2 = -4$

$\therefore$ $\alpha_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$\therefore$ solⁿ is

$$x(t) = c_1 \alpha_1 e^{m_1 t} + c_2 \alpha_2 e^{m_2 t}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{m_1 t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{m_2 t}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{m_1 t} + c_2 e^{m_2 t} \\ c_1 e^{m_1 t} \cancel{+}4c_2 e^{m_2 t} \end{bmatrix}$$

$m_1 = 2$
$m_2 = -3$

$$\left.\begin{aligned} \therefore x(t) &= c_1 e^{2t} + c_2 e^{-3t} \\ y(t) &= c_1 e^{2t} + 4c_2 e^{-3t} \end{aligned}\right] A$$

<u>Case</u> ② If roots are imaginary then say $m_1 = a+ib$, $m_2 = a-ib$

solⁿ of ⊛ is

$$x(t) = e\ e^{at}[\alpha_1 \cos bt + \alpha_2 \sin bt]$$

$\alpha_1 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$\alpha_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

# Solution and Path of an autonomous system

Consider the autonomous system

$$\circledast \begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases}$$

solution $\circledast$ $(x(t), y(t))$ represents a curve in $xy$ plane which is called path or trajectory of $\circledast$

Def① Critical Point:- Given the autonomous system

$$\frac{dx}{dt} = F(x,y), \quad \frac{dy}{dt} = G(x,y), \quad ①$$

a point $(x_0, y_0)$ at which $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$
or $F(x,y) = 0$, $G(x,y) = 0$
is called critical point of ①

~~Def~~

Def② Let $x = x(t)$, $y = y(t)$ is a solution which parametrically represents the path $C$ and let $(0,0)$ be a critical point of the autonomous system $\frac{dx}{dt} = F(x,y)$

$$\frac{dy}{dt} = G(x,y)$$

then $\boxed{\text{Path } C \text{ approaches the critical point } (0,0) \text{ as } t \to \infty}$

if $\lim_{t\to\infty} x(t) \to 0$ and $\lim_{t\to\infty} y(t) \to 0$

smly $\boxed{\text{Path } C \text{ approaches the critical point } (0,0) \text{ as } t \to -\infty \text{ if } \lim_{t\to-\infty} x(t) \to 0 \text{ and } \lim_{t\to-\infty} y(t) \to 0}$

# Types of Critical Points

**Center:-** The isolated critical point $(0,0)$ of an autonomous system is called a center if there exists a ~~neibh~~ neighbourhood of $(0,0)$ which contains a countably infinite number of closed path $C_n$ $(n=1,2,3,\cdots)$ each of which contains $(0,0)$ in its interior and which are such that the diameters of the paths approach $0$ as $n \to \infty$, but $(0,0)$ is not approached by any path either as $t \to \infty$ or $t \to -\infty$.

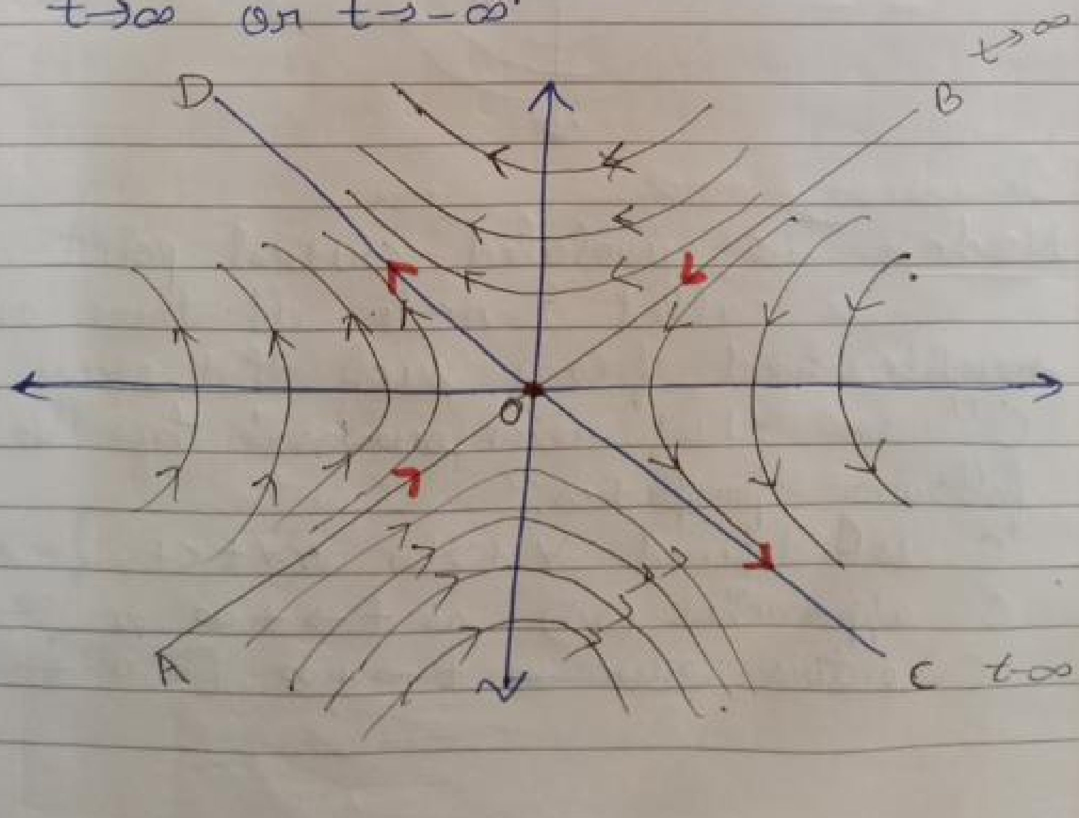

critical point $(0,0)$ is called center.

## Saddle point:-

The isolated critical point (0,0) is called a saddle point if there exist a neighbourhood of (0,0) in which the following two conditions hold;

(i) There exist two paths which approach and enter (0,0) from a pair of opposite directions as $t \to \infty$ and there exist two paths which approach and enter (0,0) from a different pair of opposite directions as $t \to -\infty$

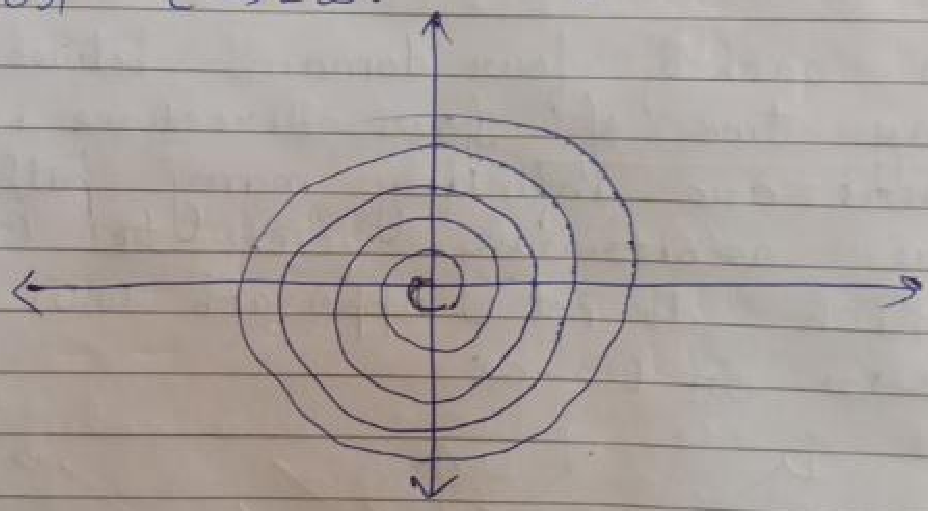
(ii) In each of four domains between, between any two of four directions in (i) there are infinitely many paths which are arbitrarily close to (0,0) but which do not approach (0,0) either as $t \to \infty$ or $t \to -\infty$.



## Spiral Point :-

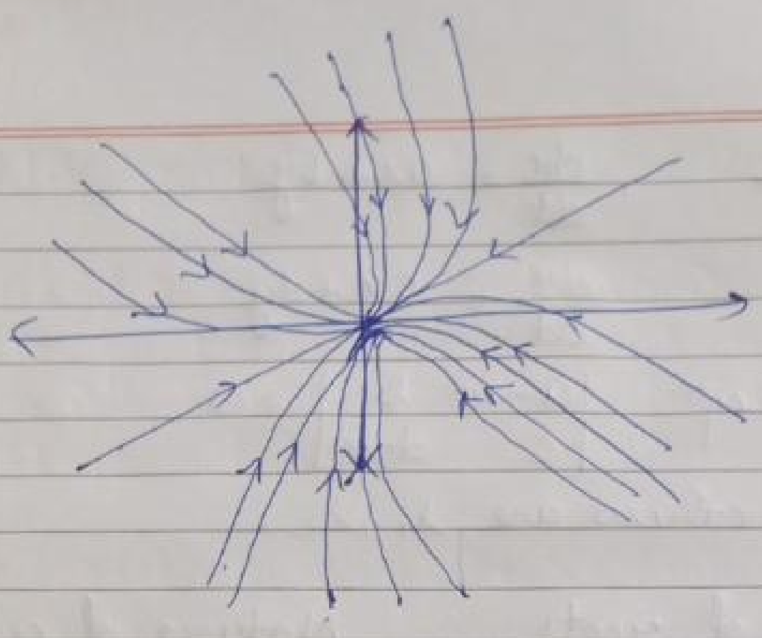
The isolated critical point (0,0) is called a spiral point, if there exists a neighbourhood of (0,0) such that every path C in this neighbourhood has the following properties:-

(i) C is defined $\forall\ t > t_0$ or $\forall\ t < t_0$ for some $t_0$
(ii) C approaches (0,0) as $t \to +\infty$ or as $t \to -\infty$
(III) C approaches (0,0) in a spiral like manner, winding around (0,0) an infinite number of times as $t \to +\infty$ or $t \to -\infty$.



Node :- The isolated critical point (0,0) is called a node if there exist a neighbourhood of (0,0) such that every path C in this neighbourhood has the following properties;

(i) C is defined $\forall\ t > t_0$ or $\forall\ t < t_0$ for some $t_0 \in$
(II) C approaches (0,0) as $t \to +\infty$ or ($t \to -\infty$)
(III) C enters (0,0) as $t \to +\infty$ [or as $t \to -\infty$



## Stability

Let $(0,0)$ is an isolated critical point of ⊛ $\frac{dx}{dt} = F(x,y)$, $\frac{dy}{dt} = G(x,y)$ — ⊛

Let $x = x(t)$, $y = y(t)$ be a solution of ⊛ defining $C$ parametrically

$D(t) = \sqrt{x(t)^2 + y(t)^2}$ = distance b/w $(0,0)$ and $(x(t), y(t))$ on $C$

∴

Critical point $(0,0)$ is stable if for every number $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. following is true

Every path $C$ for which $D(t_0) < \delta$ for $t_0$, is defined $\forall\ t \geq t_0$ and

$D(t) < \varepsilon$

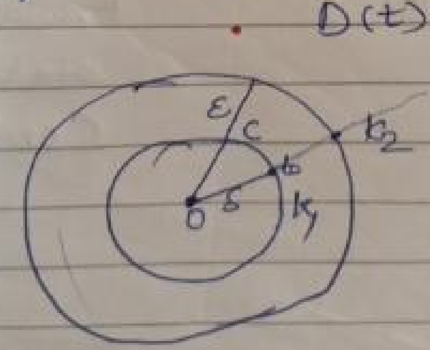



Table $\frac{dx}{dt} = ax + by$

$$\frac{dy}{dt} = cx + dy$$

char. eqⁿ $\begin{vmatrix} a-m & b \\ c & d-m \end{vmatrix} = 0$

eigen values are $\lambda_1, \lambda_2$,

| Nature of roots | Nature of critical points |
|---|---|
| 1. Real and distinct ($\lambda_1 \neq \lambda_2$) | |
| + + | unstable node |
| − − | stable node |
| + − | saddle point (unstable) |
| 2. Real and repeated | |
| + + | unstable node |
| − − | stable node |
| 3. Imaginary roots $\lambda = \alpha \pm i\beta$ | |
| $\alpha > 0$ | unstable spiral |
| $\alpha < 0$ | stable spiral |
| $\alpha = 0$ | Center |

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Examples:-

(1) $\frac{dx}{dt} = 2x - 7y$ Determine the nature of critical point.

$\frac{dy}{dt} = 3x - 8y$

$$\begin{bmatrix} 2 & -7 \\ 3 & -8 \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & -7 \\ 3 & -8-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (-6)\lambda + (-16+21) = 0$$

$$\lambda^2 + 6\lambda + 5 = 0$$

$$\lambda = -5, -1$$

$\because$ roots are real and distinct
$\therefore$ same sign with $(-)$.
$\therefore$ stable node.

(2) $\frac{dx}{dt} = 2x + 4y$

$\frac{dy}{dt} = -2x + 6y$

$$\begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

$$\lambda^2 - 8\lambda + (12+8) = 0$$

$$\lambda^2 - 8\lambda + 20 = 0$$

$$\lambda = 4 \pm 2i$$

$\alpha = 4 > 0$ $\therefore$ unstable spiral.