Melline Transform

let $f(x)$ be defined for $0 < x < \infty$.
Melline transform of $f(x)$ is a new function $\bar{f}(s)$ of the complex variable s defined by

$$\bar{f}(s) = M\{f(x)\} = \int_0^\infty x^{s-1} f(x)\,dx$$

for those values of s for which the improper integral on RHS converges.

Melline Inversion Formula.

$$M^{-1}\{\bar{f}(s)\} = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\bar{f}(s)\,ds$$

Properties of Melline Transform -

① Linear property

$$M\{a f(x) \pm b g(x)\} = a M\{f(x)\} \pm b M\{g(x)\}$$

② Change of Scale property

If $M\{f(x)\} = \bar{f}(s)$ then
$$M\{f(ax)\} = a^{-s}\bar{f}(s)$$

③ If $M\{f(x)\} = \bar{f}(s)$ then
$$M\{x^a f(x)\} = \bar{f}(s+a)$$

④ If $M\{f(x)\} = \bar{f}(s)$ then
$$M\{f(x^a)\} = \frac{1}{a}\bar{f}(s/a)$$

(5) If $M\{f(x)\} = \bar{F}(s)$ then

$$M\left\{\frac{1}{x} f(1/x)\right\} = \bar{F}(1-s)$$

(6) If $M\{f(x)\} = \bar{F}(s)$ then

$$M\{\log x \, f(x)\} = \frac{d}{ds}\bar{F}(s)$$

(7) <u>Melline transform of derivative of function</u> –

If $M\{f(x)\} = \bar{F}(s)$ then

(i) $M\{f'(x)\} = -(s-1)\bar{F}(s-1)$

(ii) $M\{f''(x)\} = (s-1)(s-2)\bar{f}(s-2)$

In general,

$$M\{f^{(n)}(x)\} = (-1)^n \frac{\Gamma s}{\Gamma s-n} \bar{f}(s-n)$$

$n = 1, 2, 3 \text{ ---}$

(8) If $M\{f(x)\} = \bar{f}(s)$ then

$$M\left\{\left(x\frac{d}{dx}\right) f(x)\right\} = -s\bar{f}(s)$$

$$M\left\{\left(x\frac{d}{dx}\right)^2 f(x)\right\} = s^2\bar{f}(s)$$

$$M\left\{\left(x\frac{d}{dx}\right)^n f(x)\right\} = (-1)^n s^n \bar{f}(s).$$

(9) If $M\{f(x)\} = \bar{F}(s)$ then

$$M\{(1+x)^{-\alpha}\} = \frac{\Gamma(\alpha-s)\,\Gamma s}{\Gamma\alpha}$$

(10) Melline Transform of Integral of a function

(I) If $M\{f(x)\} = \bar{F}(s)$ then

(i) $M\{\int_0^x f(t)\,dt\} = -\frac{1}{s}\bar{F}(s+1)$

(ii) $M\{\int_0^x dy(\int_0^y f(t)\,dt)\} = \frac{1}{s(s+1)}\bar{F}(s+2)$

In general,

$$M\{I_n f(x)\} = (-1)^n \frac{\Gamma s}{\Gamma s+n}\bar{F}(s+n)$$

where $I_n f(x) = \int_0^x I_{n-1} f(t)\,dt$

(II) $M\{\int_x^\infty f(t)\,dt\} = \frac{1}{s}\bar{F}(s+1)$

$M\{\int_0^x dy\left(\int_0^y f(t)\,dt\right)\} = \frac{1}{s(s+1)}\bar{F}(s+2)$

(11) Convolution Theorem of Melline Transform

If $\bar{F}(s)$ and $\bar{g}(s)$ are melline transforms of $f(x)$ and $g(x)$ respectively then

$$M\{f(x)\,g(x)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{F}(u)\,\bar{g}(s-u)\,du$$

Another form.

If $\bar{F}(s)$ and $\bar{g}(s)$ are Melline transforms of $f(x)$ and $g(x)$ respectively then

$$M^{-1}\{\bar{F}(s)\,\bar{g}(s)\} = \int_0^\infty \frac{1}{t} f\left(\frac{x}{t}\right) g(t)\,dt.$$