Duality in Linear Programming

Every linear programming problem whether maximization or minimization has associated with it an other single linear programming problem way from the same data. The original prograblem is called primal problem while other is called dual.

However in general either problem can be considered as trivial and the remaining as dual problem.
The format of the simplex method is such that when Trivial is solved it associated dual is also solve simultaneously.

In other words, if the optimal sol^n to one is known, the optimal sol^n of the other is readily available.

Consider the general LPP which we will call the trivial problem

$$\text{Max } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

s.t.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$$
$$x_j \ge 0,\ j = 1, 2, \ldots n$$

The dual of this problem is expressed as

$$\text{Min } z^* = b_1 y_1 + b_2 y_2 + \dots b_m y_m$$

s.t.

$$a_{11} y_1 + a_{21} y_2 + \dots + a_{m1} y_m \geq c_1$$

$$a_{12} y_1 + a_{22} y_2 + \dots + a_{m2} y_n \geq c_2$$

$$\vdots$$

$$a_{1n} y_1 + a_{2n} y_2 + \dots + a_{mn} y_n \geq c_n$$

$$y_i \geq 0 \quad , i = 1, 2, \dots m.$$

⇒ Dual problem when primal is in canonical form

Consider the following LPP in canonical form

$$\text{Max } z = \sum_{j=1}^{n} c_j x_j$$

S.t

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad , i = 1, 2, \dots m$$

$$x_j \geq 0 \ , \ j = 1, 2, \dots n$$

Dual →

$$\text{Min } z^* = \sum_{i=1}^{m} b_i y_i$$

S.t

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \ , \ j = 1, 2, \dots n$$

$$y_i \geq 0 \ , \ i = 1, 2, \dots m$$

Date
Page

Dual problem

⇒ When primal is in standard form

Consider the following LPP is in standard form

$$\text{Max } z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t } \sum_{j=1}^{n} a_{ij} x_j = b_i \quad , i = 1, 2, \cdots, m$$

$$x_j \geq 0, \ j = 1, 2, \cdots, n$$

The dual becomes

$$\text{Min } z^* = \sum_{i=1}^{m} b_i y_i$$

$$\text{s.t } \sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad , j = 1, 2, \cdots n$$

y_i unrestricted in signs for all i.

→ Find the dual of the primal problem

$$\text{Max } z = 5x_1 + 6x_2$$

s.t.

$$x_1 + 9x_2 \leq 60$$
$$2x_1 + 3x_2 \leq 45$$
$$5x_1 - 2x_2 \leq 20$$
$$x_2 \leq 30$$
$$x_1, x_2 \geq 0$$

Let y_1, y_2, y_3 and y_4 be the dual variables

$$\text{Min } z^* = 60y_1 + 45y_2 + 20y_3 + 30y_4$$

s.t.

$$y_1 + 2y_2 + 5y_3 \geq 5$$
$$9y_1 + 3y_2 - 2y_3 + y_4 \geq 6$$
$$y_j \geq 0, \ j = 1, 2, 3, 4$$

On other hand consider the primal problem

$$\text{Max } z = \sum_{j=1}^{n} c_j x_j$$

s.t.

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad , i = 1, 2, \ldots m$$

x_j unrestricted in sign

Then dual problem becomes

$$\text{Min } z^* = \sum_{i=1}^{m} b_i y_i$$

s.t. $$\sum_{i=1}^{n} a_{ij} y_i = c_j \quad , j = 1, 2, \ldots n$$

$$y_i \geq 0 \, , \; i = 1, 2, \ldots m.$$