Module – 2

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(Function of Bounded Variation)

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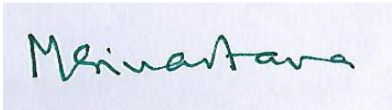

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E-Content
Function of Bounded variation
MA/M.Sc. Previous

LECTURE -1

Now today we will study about functions of bounded variation and its properties.

Definition- Functions of bounded variation consider a function f defined on [a, b]. let P = {a = x_0, x_1 - x_n = b_1| be any partition of [a, b]. The number $v(f,P) = \sum_{r=1}^{n} |\delta fr| = \sum_{r=1}^{n} |\delta f(x_r) - f(x_{r-1})|$ is called the variation of f corresponding to P.

Thus f has bounded variation on [a, b] if ∃ or there exists

$K > 0, \text{such that } V(f,P, \leq K \; \forall \; P \in P[a,b]$ ....

Where P [a, b] denotes the family of all partitions of [a,b].............

$V[f, (a,h)] = \text{Sup} \{V(f,P) : P \in P[a,h] \text{then } v[f,(a,h)] \text{is}$

write $v(f \; [a, b])$ is defined to be the total variation of f on [a, b]

defined to be the total variation of f on (a, h)

clearly if f has bounded variation on [a, b]and a < c <b, then f is on bounded variation of [a,c] and [c,b]and $V(f, [a,b]) = V(f, [a,c]) + V(f, [c,b])$..............(1)

Theorem 1: If f is of bounded variation on [a, b], then f is bounded on [a, b]

Proof: f is of bounded variation on [a, b]

$\Rightarrow \exists \; k > 0$ such that $v\{f,p\} \leq k$
$\Rightarrow v(f, [a, b]) \leq M.$

Also $|f(x) - f(a)| \leq v|f, [a,b] \leq M \; \forall \; x \in |a,b)$

Now $|f(x) - f(a) \;| \leq M \Rightarrow |f(x)| - |f(a)| \leq M$

$\Rightarrow |f\ (x)| \le M + f\ (a) \forall x \in (a, b)$

$\Rightarrow f\ (x)$ is bounded on (a, b)

Theorem 2:If the derivative f' (x) exists and is bounded in closed internal [a,b]then the function f is of bounded variation.

Proof: Let $p = \{a = x_0 \ldots x_n = b\}$ he a partition of closed internal [a, b]

Write $v\ (\delta,\ P) = \sum_{r=1}^{n} |f\ (x_n) - f\ (x_{r-1})|\ \ r = 1, 2 \ldots\ldots n$ (1)

and $V\ (f, [a,\ b]) = \sup\{v\ (f, P). P \in P[a, b]\}$ (2)

Since f' (x) is bounded in [a, b] $\exists\ K > 0$ such that $|f'\ (x)| \le K \forall x \in (a, b) -$ (3)

By Lagrange's mean value theorem

$$\frac{f\ (x_r) - f\ (x_{r-1})}{(x_r - x_{r-1})} = f'\ (\xi) \text{ where } \xi \in (a, b)$$

$$\Rightarrow f\ (x_r) - f\ (x_{r-1}) = (x_r - x_{r-1}) f'\ (\xi_r)$$

$$\therefore |f\ (x_r) - f\ (x_{r-1})| = (x_r - x_{r-1}) |f'(\xi_r)| \quad (4)$$

For $x_{r-1} < x_r$ for r = 1, 2 -n.

$$\therefore |x_r - x_{r-1}| = (x_r - x_{r-1})$$

Using (3) and (4) we get

$$|f\ (x_r) - f\ (x_{r-1})| \le (x_r - x_{r-1})\ K \quad (5)$$

Using (5) in (1) we get

$$v\ (f, P) = \sum_{r=1}^{n} (x_r - x_{r-1}) K = K (x_n - x_0) = k\ (b - a)$$

Takingsupremum of both sides and from (2) we get

$v\ (f, [a, b]) = k (b - a)$

$\Rightarrow v\ (f, [a, b])$ is finite

Consequently f is bounded.

Theorem:3 If $c \in [a, b]$ then show that f is bounded variation on [a, c] and [c, b] iff f is of bounded variation on [a, b].

Also prove that

$$V(f, [a,b] = V(f, [a,c] + V(f, [c,b])$$

Solution Let $P_1 = \{a = x_0, x_1, x_2 .....x_n = c\}$

and $P_2 = \{y_0 = c, y_1, y_2, ...y_m = b\}$

where $x_n = y_0 = c$ and $P = P_1 \cup P_2$,

Then $P = \{a = x_0, x_1 ...., x_n = c = y_0, y_1, y_2 ......y_n = b\}$

Evidently P_1, P_2, P are respectively partitions of [a, c], [c,b] and [a, b]

Step: 1. Suppose f is of bounded variation on [a,b].

To prove that f is of bounded variation on [a, c]and [c, b] both.

By assumption, V (f, [a, b] = finite = k, say.

Evidently $\sum_{r=1}^{n} |f.(x_r) - f(x_{r-1})| + \sum_{r=1}^{n} |f.(y_r) - f(y_{r-1})| \leq V(f,[a,b]) = k$

Write $S_1 = \sum_{r=1}^{n} |f.(x_r) - f(x_{r-1})|$, $S_2 = \sum_{r=1}^{m} |f.(y_r) - f(y_{r-1})|$

Then $S_1, S_2 > 0$ and $S_1 + S_2 \leq k$, $S_2 < S_1 + S_2 \leq k$

or $S_1 \leq k, \quad S_2 \leq k$

Taking supremum over all partitions P_1 and P_2

$\therefore \quad V(f, [a,c]) \leq k, \; V(f,[c,b] \leq k$

This $\Rightarrow$ f is of bounded variation on [a, c] and [c,b] both.

Step II. Let f be of bounded variation on [a, c] and [c, b] respectively. Then

$$V(f, [a,c]) = k_1 = \text{finite},$$
$$V(f, [c,b]) = k_2 = \text{finite},$$

Aim f is of bounded variation on [a, b].

Let P = {a = z_0, z_1, z_2......, z_n = c, z_{n+1}, ...,z_m = b } be a partition of [a, b]

Then

$$\sum_{r=1}^{n}|f.(z_r)-f(z_{r-1})| = \sum_{r=1}^{n}|f.(z_r)-f(z_{r-1})| + \sum_{r=n+1}^{m}|f.(z_r)-f(z_{r-1})|$$

$$\leq V(f,[a, c]) + V(f,[c, b] = k_1 + k_2 = \text{finite}$$

or $\sum_{r=1}^{n}|f.(z_r)-f(z_{r-1})| \leq k_1 + k_2 = \text{finite}$ (1)

This $\Rightarrow$ f is of bounded variation on [a, b].

Also, by (1)

$V(f,[a, b]) \leq V(f,[a, c]) + V(f,[c, b])$ (2)

Step III: f is of bounded variation on [a, c].

$\Rightarrow$given$\varepsilon > 0$ $\exists$ a partition P_1 = {a = x_0, x point x_n = C}

Such that $\sum_{r=1}^{n}|f.(x_r)-f(x_{r-1})|, > V(f,[a,c]) - \frac{\varepsilon}{2}$ (3)

Similarly f is of bounded variation on [c, b]

gives $\sum_{r=1}^{n}|f.(y_r)-f(y_{r-1})|, > V(f,[c,b]) - \frac{\varepsilon}{2}$ (4)

Adding (3)and (4), we get

$$\sum_{r=1}^{n}|f.(x_r)-f(x_{r-1})|, + \sum_{r=1}^{m}|f.(y_r)-f(y_{r-1})|$$

$$> V(f,[a, c]) + V(f,[c, b]) - \varepsilon$$

or $V(f,[a, b]) > V(f,[a, c]) + V(f,[c, b]) - \varepsilon$

Making $\varepsilon > 0$, we get

$V(f,[a, b]) \geq V(f,[a, c]) + V(f,[c, b])$ (5)

(2) and (5) $\Rightarrow$ $V(f,[a, b]) = V(f,[a, c]) + V(f,[c, b])$ where $a \leq c \leq b$.

$\therefore$ $V(f, [a, b]) = k_1 + k_2$ = finite

$\Rightarrow$f is of bounded variation in [a,b]

LECTURE -2

we will study about variation function and its properties and theorem based on it.

Def. Variation Function, Let f be a function of bounded variation on [a, b] and $x \in [a, b]$. The total variation of f on [a, c] is denoted by V (f, [a, x) which is clearly a function of x.

Write $v_j(x) = v(f, [a, x])$

Then $v_j(x)$ is defined as variation function or total variation function of f. Sometimes we also write$v_j(x) = v(x)$.

Let $x_1, x_2, \in [a, b]$ be arbitrary s.t. $x_1 < x_2$.

Since $V(f,[a, b]) = V(f,[a, c]) + V(f,[c, b])$

and so $V(f,[a, b]) - V(f,[a, c]) = V(f,[c, b])$

Hence $V(f,[a, x_2]) - V(f,[a, x_1]) = V(f,[x_1,X_2])$ (1)

or $v(x_2) - v(x_1) = V(f,[x_1,X_2])$ (2)

Since $0 \leq |f(x_2) - f(x_1)| \leq V(f,[x_1,X_2])$

Using (2), we get $v(x_2) - v(x_1) \geq 0$

or $v(x_2) \geq v(x_1)$

Thus $x_1 < x_2 \Rightarrow v(x_1) \leq v(x_2)$

This $\Rightarrow$ v(x) is monotonic non-decreasing function on [a, b].

Theorem 4: The variation function v(x) of a function of bounded variation is continuous iff f is a continuous function.

Proof. Let $f: [a,b] \to R$ be a map. Let $x, c \in [a,b]$ be arbitrary but $x < c$.

We have $v(x) = V(f, [a, x])$ (1)

$0 \leq |f(x) - f(c)| \leq V(f,[x, c])$ (2)

and $V(f, [x, c] = V(f, [a, c] - V(f, [a, x])$ (3)

By (2) and (3), we get

$$0 \leq |f(x) - f(c)| \leq V(f,[a, c]) - V(f, [a, x])$$

$= v(c) - v(x)$

or $\quad 0 \leq |f(x) - f(c)| \leq v(c) - v(x)$ (4)

$x < c \Rightarrow v(x) < v(c)$ as $v(x)$ is monotonic increasing function .

(4) $\quad \Rightarrow 0 \leq |f(x) - f(c)| \leq v(c) - v(x)$ (4')

Let f be a function of bounded variation so that

$v(c), v(x) \leq$ finite number (5)

Step 1. Let v(x) be continuous on [a, b] (6)

Aim: f(x) is continuous on [a, b].

For this we show that f(x) is continuous at x = c.

By (6) given $\varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|x - c| < \delta \Rightarrow |v(x) - v(c)| < \varepsilon \qquad (7)$$

Using (4) in (7) we get

$$|f(x) - f(c)| < \varepsilon \text{ for } |x - c| < \delta$$

$\therefore$ f(x) is continues at x = c.

Step II. Let f (x) be continuous on [a, b]

Aim.v(x) is continuous on [a, b] .

By assumption, given $\varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \frac{\varepsilon}{2} \qquad (8)$$

$\because$ f is bounded variation

$\therefore \exists$ partition $p = [c \quad x_0, x_1, \ldots\ldots\ldots, x_n = b\}$ of $[c, b]$

s.t. $\displaystyle\sum_{r=1}^{n} |f.(x_r) - f(x_{r-1})|, > V(f, [c, b]) - \frac{\varepsilon}{2}$ (9)

Further suppose that length of first sub-interval $x_1 - c$ is less than δ. By doing this change, (9) will remain uneffected.

$\Rightarrow \quad 0 < |x_1 - c| = x_1 - c < \delta$

and by (8) $\left| f(x_1) - f(c) < \frac{\varepsilon}{2} \right.$ (10)

and by (8) and (9)

$$\left| f(x_1) - f(c) \right| . + \sum_{r=2}^{n} \left| f.(x_r) - f(x_{r-1}) \right|, > V(f,[c,b]) - \frac{\varepsilon}{2}$$

$$\text{or } V(f,[c,b]) - \sum_{r=2}^{n} \left| f.(x_r) - f(x_{r-1}) \right|, < \left| f(x_1) - f(c) \right| + \frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{by (10).}$$

or $V(f, [c,b]) - V(f, [x_1, b]) < \varepsilon$

or $V(f, [c, x_1]) < \varepsilon$ (11)

But $V(f, [a, b] = V(f, [a, c]) + V(f, [c, b])$

$V(f, [a, b] - V(f, [a, c]) = V(f, [c, b])$

or $V(f, [a, x_1] - V(f, [a, c]) = V(f, [c, x_1]) < \varepsilon$, by (11)

or $V(f, [a, x_1] - V(f, [a, c]) < \varepsilon$

or $v(x_1) - v(c) < \varepsilon$

or $| v(x_1) - v(c)| < \varepsilon$ v(x) is an increasing function.

for $| x_1 - c| < \delta$

This $\Rightarrow$ v(x) is continuous at x = c.

Theorem 5: A monotonic function is a function of bounded variation.

or

Let be monotonic function and bounded on [a, b]. Then f is bounded variation on [a, b] and $v(f) = | f(b) - f(a)|$

Proof: let f : [a, b] $\rightarrow$ R be a monotonic function.

Let P = {a = x_0,, x_1, x_2,......, x_n = b} be a partition of [a, b]

Then $$v(f,P) = \sum_{r=1}^{n} \left| f(x_r) - f(x_{r-1}) \right| \quad (1)$$

Case 1: When f is monotonic increasing function

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \Rightarrow f(x_2) - f(x_1) > 0$

$\Rightarrow |f(x_2) - f(x_1)| = f(x_2) - f(x_1)$

Now (1) becomes

$$v(f, P) = \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})]$$

$= [f(x_1) - f(x_0)] + [f(x_2) - f(x_1)] + \ldots + [f(x_n) - f(x_{n-1})]$

$= f(x_n) - f(x_0) = f(b) - f(a)$

or $\quad v(f, P) = |f(b) - f(a)|$

as $\quad a < b \Rightarrow f(a) < f(b)$. (2)

Case II: When f is monotonic decreasing.

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \Rightarrow f(x_1) - f(x_2) > 0$

Now (1) becomes

$$v(f, P) = \sum_{r=1}^{n} [f(x_{r-1}) - f(x_r)$$

$= [f(x_0) - f(x_1)] + [f(x_1) - f(x_2)] + \ldots + [f(x_{n+1}) - f(x_n)]$

$= f(x_0) - f(x_n) = f(a) - f(b)$

as $\quad a < b \Rightarrow f(a) > f(b) \Rightarrow f(a) - f(b) > 0$

$\therefore \quad v(f, P) = |f(a) - f(b)|$ (3)

By (2) and (3),, we have, in either case,

$v(f, P) = |f(b) - f(a)| =$ finite

This $\Rightarrow$ f is of bounded variation

LECTURE -3

We will study some algebraic properties of bounded variation

Theorem 6: If f and g are functions of bounded variation on [a, b], then their sum f + g, product f g and difference f-g are functions of bounded variation on [a, b]

Proof: Let p = {a = x_0, x_1, x_2, ...x_n = b } be a partition of [a, b].

$$v(f,P)=\sum_{r=1}^{n}|f(x_r)-f(x_{r-1})|$$

$$v(g,P)=\sum_{r=1}^{n}|g(x_r)-g(x_{r-1})|$$

f and g are of bounded variation on [a, b]

$\Rightarrow \exists\ k_1, k_2 > 0$ s.t. v (f, P) < k_1, v (g, P) < k_2,

Also they must be bounded on [a, b] and so $\exists\ m_1, m_2 > 0$s.t.

$$|f(x)|<m_1, |g(x)|\le m_2 \ \ \forall x\in[a,b]$$

Step1: To prove that f + g is of bounded variation

$$v(f+g,P)=\sum_{r=1}^{n}|(f+g)(x_r)-(f+g)(x_{r-1})|$$

$$=\sum|\{f(x_r)-f(x_{r-1})\}+\{g(x_r)-(x_{r-1})|$$

$$\le\sum|f(x_r)-f(x_{r-1})\}+\sum\ g(x_r)-g(x_{r-1})|$$

$$=v(f,P)+v(g,P)<k_1+k_2$$

or, v (f + g, P) < $k_1 + k_2$

This $\Rightarrow$ f + g is of bounded variation.

Note. Here we have V (f+g. I) ≤ V (f, I) + V (g,I)where I = [a, b].

Step II: To prove that fg is of bounded variation on [a, b].

$$v(fg,P)=\sum_{r=1}^{n}|(fg)(x_r)-(fg)(x_{r-1})|$$

$$=\sum|\{f(x_r)g(x_r)\}-f(x_{r-1})g(x_{r-1})|$$

$$= \sum \left| f(x_r)\{g(x_r) - g(x_{r-1}) - g(x_{r-1})\} + g(x_{r-1})\{f(x_r) - f(x_{r-1})\} \right|$$

$$\leq \sum \left| f(x_r) \right| \; \left| g(x_r) - g(x_{r-1}) + \sum \right| \; g(x_{r-1}) \left| \; \right| f(x_r) - (x_{r-1}) \left| \right.$$

$$\leq m_1 \sum_{r=1}^{n} \left| g(x_r) - g(x_{r-1} \right| + m_2 \sum_{r=1}^{n} \left| f(x_r) - f(x_{r-1} \right|$$

$$= m_1 v(g, P) + m_2 \; v(f, P)$$

$$< m_1 k_2 + m_2 k_1$$

or $\quad v(fg, P) < m_1 k_2 + m_2 k_1 = \text{finite}$

This $\Rightarrow$fg is of bounded variation

Step III: To prove that f - g is of bounded variation.

$$v(f - g, P) = \sum_{r=1}^{n} \left| (f - g)(x_r) - (f - g)(x_{r-1}) \right|$$

$$= \sum \left| \{f(x_r) - g(x_r)\} - \{f(x_{r-1}) - g(x_{r-1})\} \right|$$

$$= \sum \{ \left| f(x_r) - f(x_{r-1})\} - \{g(x_r) - g(x_{r-1})\} \right|$$

$$\leq \sum \left| f(x_r) - f(x_r - 1) \; \right| + \sum \left| \; g(x_r) - g(x_{r-1}) \right|$$

$$= v(f, P) + v(g, P) < k_1 + k_2$$

or $\quad v(f - g, P) < k_1 + k_2 = \text{finite}$

This $\Rightarrow$ f-g is of bounded variation

Note: Here we have the following results.

$$V(f + g, I) \leq V(f, I) + V(g, I)$$

$$V(f - g, I) \leq V(f, I) + V(g, I)$$

$$V(fg, I) \leq m_1 V(g, I) + m_{2V}(f, I)$$

Where I = [a, b], $|f(x)| \leq m_1$, $|g(x)| < m_2$

Theorem 7: If a function f is of bounded variation of [a, b] and if $\exists$k> 0 s.t.

$|f(x)| \geq k \;\; \forall \;\; x \in [a, b]$, then $\frac{1}{f}$ is also of bounded variation.

Proof, Let P = {a = x_0, x_1, x_2......, x_n = b} be a partition of [a, b]. Then

$$v\left(\frac{1}{f},P\right) = \sum_{r=1}^{n}\left|\frac{1}{f(x_r)} - \frac{1}{f(x_{r-1})}\right|$$

$$= \sum\left|\frac{f(x_{r-1}) - f(x_r)}{f(x_r)\ f(x_{r-1})}\right|$$

$$= \sum\frac{|f(x_e) - f(x_{-1})|}{|f(x_r)|.|f(x_{r-1})|}$$

$$\leq \frac{1}{k^2}\sum|f(x_r) - f(x_{r-1})| = \frac{1}{k^2}v(f,P)$$

as $\quad |f(x)| \geq k \Rightarrow \frac{1}{|f(x)|} \leq \frac{1}{k}$

or $\quad v\left(\frac{1}{f},P\right) = \frac{1}{k^2}v(f,P) \qquad (1)$

Further f is of bounded variation.

$\therefore \qquad v(f,P) < k_1$, where $k_1 > 0$.

$\therefore$ By (2) $\qquad v\left(\frac{1}{f},P\right) < \frac{k_1}{k^2}$ = finite number

$\therefore \quad \frac{1}{f}$ is of bounded variation.

Theorem 8: (Jordan Theorem): A function of bounded variation is expressible as a difference of two monotone increasing functions.

Proof: Let P = {a = x_0, x_1, x_2, ...,x_n = b} be a partition of [a, b]. Then

$$v(f,P) = \sum_{r=1}^{n}|f(x_r) - f(x_{r-1})|$$

$$v(f,[a,b]) = \sup\{v(f,P) : P \in P[a,b]$$

Let x $\in$[a, b] be arbitrary. Then

v(x) = V (f, [a, x]) is called variation function. We define

$$P(x) = \frac{1}{2}[v(x) + f(x)] \tag{1}$$

$$q(x) = \frac{1}{2}[v(x) - f(x)] \tag{2}$$

v(x) is a monotonic increasing function.

Evidently $x_1 < x_2 < x_3$ ....

I. $$P(x_2) - P(x_1) = \frac{1}{2}[v(x_2) + f(x_2)] - \frac{1}{2}[v(x_1) + f(x_1)]$$

or $$P(x_2) - P(x_1) = \frac{1}{2}[v(x_2) - v(x_2)] + \frac{1}{2}[f(x_2) - f(x_1)]$$

But $$v(x_2) - v(x_1) = v(f, [a, x_2]) - [v(f, [a, x_1])$$

$$= \quad v(f, [x_1, x_2]) \geq |f(x_2) - f(x_1)|$$

$\therefore$ $$P(x_2) - P(x_1) \geq \frac{1}{2}[v(x_2) - v(x_1)] + \frac{1}{2}\{f(x_2) - f(x_1)\} \geq 0$$

or, $P(x_2) - P(x_1) \geq 0$ or $P(x_2) \geq P(x_1)$

Thus $x_1 < x_2 \Rightarrow P(x_1) \leq P(x_2)$

This proves that P (x) is an increasing function.

II. $$q(x_2) - q(x_1) = \frac{1}{2}[v(x_2) - f(x_2)] - \frac{1}{2}[v(x_1) - f(x_1)]$$

$$= \frac{1}{2}[v(x_2) - v(x_1)] - \frac{1}{2}[f(x_2) - f(x_1)]$$

$$= \frac{1}{2}v(f[x_1, x_2]) - \frac{1}{2}[f(x_2) - f(x_1)]$$

$$\geq \frac{1}{2}|f(x_2) - f(x_1)| - \frac{1}{2}\{f(x_2) - f(x_1)\} \geq 0$$

or $q(x_2) - q(x_1) \geq 0$

or $q(x_2) \geq q(x_1)$

or $x_1 < x_2 \Rightarrow q(x_1) \leq q(x_2)$

This $\Rightarrow$ q (x) is monotonic increasing function.

(1) - (2) gives P (x) - q (x) = f (x)

or $\qquad f(x) = P(x) - q(x)$

This $\Rightarrow$ f(x) is expressible as a difference of two monotonic increasing functions.

Theorem 9: If is of bounded variation on *[a, b],* then *V =P + N* and *P-N= f(b)-f(a),* where *V, P,N* respectively denote total, positive and negative variations of on *[a, b].*

Or

$$T_a^b = P_a^b + N_a^b \text{ and } P_a^b - N_a^b = f(b) - f(a).$$

Proof. Let $\quad V = \sum_{r=1}^{n-1} |f(x_{r+1}) - f(x_r)|.$

Here the closed interval [a, b] is divided by means of points

$$a = x_0 < x_1 < x_2 < \ldots . < x_n = b.$$

Let p be the sum of those differences $f(x_{r+1})- f(x_r)$ which are positive. - n that the sum of those differences which are negative. Evidently

$$\nu = P + n, \; f(b) - f(a) = p - n.$$

From which we get

$$\nu + f(b) - f(a) = 2p.$$

and $\qquad \nu - f(b) + f(a) = 2n.$

i.e. $\qquad \nu = 2p + f(a) - f(b) \qquad (1)$

and $\qquad \nu = 2n + f(b) - f(a) \qquad (2)$

Set $P = \sup p$, $N = \sup n$, $V = \sup \nu$

where we take the suprema over all possible sub-division of [a, b]. Taking suprema in (1) and (2).

$$V = 2p + f(a) - f(b) \qquad (3)$$

$$V = 2N + f(b) - f(a) \qquad (4)$$

upon addition, $\qquad 2V = 2P + 2N$

or $$V = P + N \tag{5}$$

(3) - (4) gives

$$0 = 2(P - N) + [f(a) - f(b)$$

or $$f(b) - f(a) = P - N \tag{6}$$

(5) and (6) ⇒ required results.

Theorem 10: To prove that a function f is of bounded variation if and only if it is expressible as a difference of two monotonic functions both non-increasing or both non-decreasing.

Proof: Let a function f be defined and finite on a closed interval [a, b] so that f (a) and f(b) are finite numbers. We shall show that

(i) if f is of bounded variation then it is represent able as a difference of two monotonic increasing functions.

(ii) If f = g-h, where g and h both are monotonic increasing functions, then f is of bounded variation.

(i) Let f be of bounded variation. Divide the interval [a, b] by means of points

$$a = x_0 < x_1 < x_2 < \ldots\ldots < x_n = b.$$

Let $$\nu = \sum_{r=0}^{n-1} |f(x_{r+1} - f(x_r)|, \sup \nu = V$$

Let P be the sum of those differences f $(x_{r+1}$-f (x_r) which are positive, - n that the sum of those differences which are negative

Evidently $$\nu = p + n, f(b) - f(a) = p - n$$

Solving for p and n, we get

$$\nu + f(b) - f(a) = 2p$$

$$\nu - f(b) + f(a) = 2n$$

$$\Rightarrow \quad \left.\begin{aligned} \nu &= 2p + f(a) - f(b) \\ \text{and} \quad \nu &= 2n + f(b) - f(a) \end{aligned}\right] \quad (1)$$

Set $P = \sup p$, $N = \sup n$, $V = \sup \nu$

where the suprema is taken over all possible subdivisions of [a, b].

Taking suprema in (1), we obtain

$$V = 2P + f(a) - f(b) \qquad (2)$$

$$V = 2N + f(b) - f(a) \qquad (3)$$

Further we suppose that V (x), P (x), N (x) respectively denote total variation, positive and negative variations of f in the interval [a, x], where $x \leq b$. With the help of (2) and (3),

$$V = 2P(x) + f(a) - f(x) \qquad (4)$$

$$V = 2N(x) + f(x) - f(a) \qquad (5)$$

(4)- (5) gives

$$0 = 2[P(x) - N(x)] + 2[f(a) - f(x)]$$

or $$f(x) = P(x) - N(x) + f(a)$$

Taking f (a) + P (x) = P' (x) , we get

$$f(x) = P'(x) - N(x) \qquad (6)$$

It is easy to verify that P (x) and N (x) both are monotonic increasing functions. P (x) is an increasing function implies that p (x) is also increasing function.

Now the required result at once follows from (6).

(ii) Let f=g-h, where g and h both are increasing functions, For any mode of sub-division of [a, b],

$$V(f) = \sum_{r=0}^{n-1} \left| f(x_{r+1}) - f(x_r) \right|$$

where $a = x_0 < x_1 < x_2 < \ldots < x_n = b$

$$\left| f(x_{r+1}) - f(x_r) \right| = \left| [g(x_{r+1}) - h(x_{r+1})] - [g(x_r) - g(x_r)] \right|$$

$$= \left| [g(x_{r+1}) - g(x_r)] - [h(x_{r+1}) - h(x_r)] \right|$$

$$= \left| g(x_{r+1}) - g(x_r) \right| - \left| h(x_{r+1}) - h(x_r) \right|$$

$$\leq \left| g(x_{r+1}) - g(x_r) \right| + \left| h(x_{r+1}) - h(x_r) \right|$$

$$= \left| g(x_{r+1}) - g(x_r) \right| + \left| h(x_{r+1}) - h(x_r) \right|.$$

For *g* and *h* both are monotonic increasing functions.

$$\therefore \qquad \nu = \sum_{r=0}^{n-1} \left| f(x_{r+1}) - f(x_r) \right| \leq \sum_{r=0}^{n-1} [f(x_{r+1}) - f(x_r)] + \sum_{r=0}^{n-1} [h(x_{r+1}) - h(x_r)]$$

$$= g(b) - g(a) + h(b) - h(a)$$

$$= \text{a finite number.}$$

[For *f* is finite valued $\Rightarrow$ *g* and *h* both are finite valued].

$$\therefore \qquad \nu < V(f) \leq \text{ a finite number}$$

This $\Rightarrow$ *f* is of bounded variation.

LECTURE -4

We will study some problems related to absolute continuous function

Theorem11: Every absolutely continuous function is of bounded variation.

Proof. Let f be an absolutely continuous function on a closed interval [a, b] so that we can select a δ>s.t.

$$\sum_{k=1}^{n} \left| f(b_{k}) - f(a_k) \right| < 1 \text{ whenever} \sum_{k=1}^{n} (b_k - a_k) < \delta$$

for all numbers a_1, b_1, a_2, b_2,......, a_n, b_n

where $a = a_1 < b_1 \leq a_2 < b_2 \leq .... \leq a_n < b_n = b$.

Again divide the closed interval [a, b] by means of points

$$a = c_0 < c_1 < c_2 < ...... < c_{n_0} = b$$

in n_0 parts s.t. $c_{k+1} - c_k < \delta$.

Consequently for any subdivision of $[c_k, c_{k+1}]$

$$\sum_{\iota} \left| f(x_{\iota+1}) - f(x_\iota) \right| \leq 1 \text{ where } x_{\iota+1}, x_\iota \in [c_k \, c_{k+1}]$$

i.e. $\quad V_{c_k}^{c_{k+1}}(f) \leq 1.$

It follows that

$$V_a^b (f) = V_{c_0}^{c_1}(f) + V_{c_1}^{c_2}(f) + ... + V_{c_{0-1}}^{c_{n0}} (f)$$

$$\leq 1 + 1 + 1 + ... = n_0$$

i.e. $\quad V_a^b (f) < \infty$

Consequently *f* is of bounded variation.

Theorem 12: If *f (x)* and *g (x)* are absolutely continuous functions, then their sum, difference and product are also absolutely continuous functions. Further if *g (x)* does not vonish for any x, then the quotient $\frac{f(x)}{g(x)}$ is also absolutely continuous.

Proof. Let *f (x)* and *g(x)* be absolutely continuous functions over the closed interval [a, b] so that

Given $\varepsilon > 0, \exists\, \delta > 0$ s.t.

$$\sum_{k=1}^{n} \left| f(b_k) - g(a_k) \right| < \varepsilon \text{ and } \sum_{k=1}^{n} \left| g(b_k) - g(a_k) \right| < \varepsilon$$

whenever $\sum_{k=1}^{n} (b_k - a\ ) < \delta$

$$\forall\ a_1,\ b_1, a_2,\ b_2, \ldots\ldots, a_n,\ b_n, \text{s.t.}$$

$$a_1 < b_1 \le a_2 < b_2 \le \ldots\ldots \le a_n < b_n$$

Step (i): To prove that *f(x)±g(x)* is absolutely continuous over [a, b].

$$\sum_{k=1}^{n} \left| f(b_k) \pm g(b_k) - [f(a_k) \pm g(a_k)] \right|$$

$$\sum_{k=1}^{n} \left| f(b_k) - f(a_k) \right| + \sum_{k=1}^{n} \left| g(b_k) - g(a_k) \right|$$

$$< \varepsilon + \varepsilon$$

Finally

$$\sum_{k=1}^{n} \left| f(b_k) \pm g(b_k) - [f(a_k) \pm g(a_k)] \right| < 2\varepsilon$$

From this the required result follows.

Step (ii): to prove that f(x) g(x) is absolutely continuous over [a, b]

$$\sum_{k=1}^{n} \left| f(b_k) g(b_k) - f(a_k) g(a_k) \right|$$

$$= \sum_{k=1}^{n} \left| f(b_k)[g(b_k) - g(a_k)] + g(a_k)[f(b_k) - f(a_k)] \right|$$

$$\le \sum_{k=1}^{n} \left| f(b_k) \right| . \left| g(b_k) - g(a_k) \right| + \sum_{k=1}^{n} \left| g(a_k) \right| \ \left| f(b_k) - f(a_k) \right|$$

or $\sum_{k=1}^{n}\left| f(b_k)g(b_k)-f(a_k)g(a_k)\right|$

$$\le \sum_{k=1}^{n}\left| f(b_k)\right|.\left| g(b_k)-g(a_k)\right| + \sum_{k=1}^{n}\left| g(a_k)\right| \left| f(b_k)-f(a_k)\right| \text{.........(1)}$$

We know that

Absolutes continuity ⇒continuity

⇒boundednes

⇒*f(x)* and *g(x)* are bounded in [a, b]

⇒$f(x) \le M_1$, $g(x) \le M_2$, $\forall x \in [a, b]$.

M_1 and M_2 are upper bounds of *f(x)* and *g(x)* respectively in [a, b].

In this event (1) takes the form

$$\sum_{k=1}^{n}\left| f(b_k)[g(b_k)-f(a_k)] \quad g(a_k)\right| < \varepsilon\left(\left|M_1\right|+\left|M_2\right|\right).$$

Setting $\varepsilon\left(\left|M_1\right|+\left|M_2\right|\right)=\varepsilon'$, we get

$$\sum_{k=1}^{n}\left| f(b_k)[g(b_k)-f(a_k)]+g(a_k)\right| < \varepsilon..$$

From this the required result follows.

Step (iii): Let g (x) vanish no where in [a, b] so that $\exists \delta>0$ s. t.$\left| g(x)\right| \ge \sigma \forall x \in$ [a, b].

To prove that $\frac{f(x)}{g(x)}$ is absolutely continuous over [a, b].

$$\sum_{k=1}^{n}\left|\frac{1}{g(b_k)}-\frac{1}{g(a_k)}\right| = \sum_{k=1}^{n}\frac{\left|g(b_k)-g(a_k)\right|}{\left|g(b_k g(a_k)\right|} < \frac{\varepsilon}{\sigma^2}$$

Setting $\frac{\varepsilon}{\sigma^2}=\varepsilon'$ we get

$$\sum_{k=1}^{n}\left|\frac{1}{g(b_k)}-\frac{1}{g(a_k)}\right| < \varepsilon',$$

This proves that $\frac{1}{g(x)}$ is absolutely continuous over [a, b].

By step (ii), $f(x)\frac{1}{g(x)} = \frac{f(x)}{g(x)}$ is absolutely continuous over [a, b]. This concludes the problem.

Theorem 13: (Integration by parts). *Let f and g be functions of bounded variation on [a, b] and let f be continuous on [a, b]. Then*

$$\int_a^b f dg = [f(x) g(x)]_a^b - \int_a^b g df$$

$$= f(b)\, g(b) - f(a) g(a) - \int_a^b g df$$

Proof. Let P= {a $=x_0, x_1, \ldots\ldots, x_n =$b} be a partition of [a, b] and Q = {a=ξ_0, ξ_1, $\xi_2, \ldots\ldots, \xi_n$ = b} be an intermediate partition of *P*.

so that $x_{r-1} \le \xi_r \le x_r$, for r = 1, 2......, n.

From the sum $S(f, g, P) = \sum_{r=1}^{n} f(\xi_r)\delta g_r$ (1)

Evidently when $\|P\| = \max \delta_r = \max (x_r - x_{r-1}) \to 0$, as $n \to \infty$ then

$$S(f, g, P) = \int_a^b f\, dg$$

By (1) $S(f, g, P) = \sum_{r=1}^{n} f(\xi_r)[g(x_r) - g(x_{r-1})]$

$$= f(\xi_1)\{g(x_1) - g(x_0)\} + f(\xi_2)\{g(x_2) - g(x_1)\} + \ldots\ldots + f(\xi_n)\{g(x_n) - g(x_{n-1})\}$$

[adding and subtracting $f(\xi_0)\, g(x_0)$ and re-arranging the terms]

$$= -(\xi_0) g(x_0) - [g(x_0)\{f(\xi_1) - f(\xi_0)\} + g(x_1)\{f(\xi_2) - f(\xi_1)\} + \ldots.$$

$$+ g(x_{n-1})\{f(\xi_n) - f(\xi_{n-1})\}\ ] + f(\xi_n) g(x_n)$$

$$= [f(\xi_n) g(x_n) - f(\xi_0) g(x_0)\ \} - \sum_{r=1}^{n} g(x_{r-1})\{f(\xi_r) - f(\xi_{r-1})\}$$

$$= [f(x) g(x)]_a^b - \sum_{r=1}^{n} g(x_{r-1})\{f(\xi_r) - f(\xi_{r-1})\}$$

$= [f(x)g(x)]_a^b - S(f, g, P, Q)$

Making $\|P\| \to 0$ and so also $\|Q\| \to 0$, we get

$$\int_a^b fdg = [f(x)g(x)]_a^b - \int_a^b fdg$$

Theorem 14: Second Mean Value Theorem, Let *f* be monotonic and g be real valued continuous and of bounded variation on [a, b]. Then $\exists \xi \in [a, b]$ such that

$$\int_a^b fdg = f(a)\{g(\xi) - g(a)] + f(b)\,[g(b) - g(\xi)].$$

Proof: By Theorem integration by parts.,

$$\int_a^b fdg = f(b)g(b) - f(a)g(a) - \int_a^b gdf \qquad (1)$$

By the first mean value theorem $\exists \xi \in$ [a, b] such that

$$\int_a^b gdf = g(\xi)[f(b) - f(a)],$$

Using this in (1), we get

$$\int_a^b gdf = f(b)g(b) - f(a)g(a) - g(\xi)[f(b) - f(a)]$$

$$= f(a)[g(\xi) - g(a)] + f(b)[g(b) - g(\xi)].$$

13.7 Change of Variable:

Theorem15: Let *f* and ϕ be continuous on [a, b] and let ϕ be increasing on [a, b]. If F is inverse function of ϕ, then

$$\int_a^b f(x)dx = \int_{\phi(a)}^{\phi(b)} .f\,[F(y).d[F(y)].$$

Proof: Let $P = \{a = x_0, x_1, x_2, \ldots\ldots x_n = b\}$ be a partition of [a, b]. Let $y_r = \phi(x_r)$, so that $x_r = F(y_r)$, as F is inverse function of ϕ. Consider the partition Q defined by $Q = \{y_0 = \phi(a), y_1, \ldots\ldots y_n = \phi(b)\}$ of [a, b]. We put $h(y) = f[F(y)]$.

Since ϕ is continuous on [a, b] it is uniformly continuous on [a, b]. Also by definition $\|Q\| \to 0$, if $\|P\| \to 0$. Further

$$\lim_{\|P\|\to 0} \sum_{r=1}^{n} f(x_r)(x_r - x_{r-1}) = \int_a^b fdx$$

and $$\lim_{\|Q\|\to 0} \sum_{r=1}^{n} h(y_r)[F(y_r) - F(y_{r-1}) = \int_{y_0}^{y_n} hdF$$

$$= \int_{\phi(a)}^{\phi(b)} f(F)dF. \qquad (1)$$

Now $x_r = F(y_r)$ and $h(y) = f[F(y)$ give

$$\sum_{r=1}^{n} f(x_r)(x_r - x_{r-1}) = \sum_{r=1}^{n} f\{F(y_r)]\{F(y_r) - F(y_{r-1})\}.$$

Making $\|P\| \to 0$ and so $\|Q\| \to 0$, we get

$$\int_a^b fdx = \int_{y_0}^{y_n} h(y)\, d[F(y)] \int_{\phi(a)}^{\phi(b)} f[F(y)]d\{F(y)], \text{by} \quad (1)$$