② **Vector Product** → Vector product of two vector $\vec{A}$ and $\vec{B}$ written as $\vec{A}\times\vec{B}$ ($\vec{A}$ cross $\vec{B}$) because this is called cross product.

$$\vec{A}\times\vec{B} = |A||B| \sin\theta\, \hat{n}$$

Here $\hat{n}$ is unit vector in the direction of $\vec{A}\times\vec{B}$

**Properties of Vector product**.

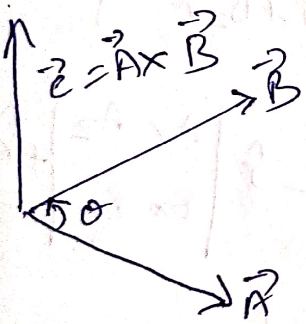
① Vector product is not commutative. it is anti commutative

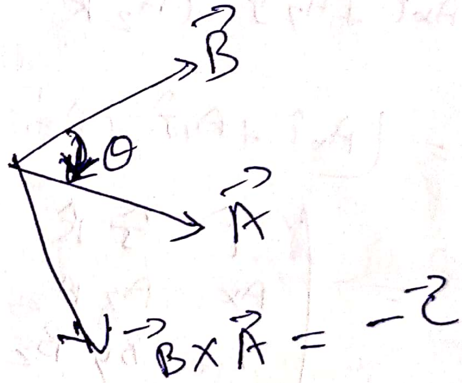
$$\vec{A}\times\vec{B} = AB\sin\theta$$
$$\vec{B}\times\vec{A} = BA\sin\theta$$

$$|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}| \qquad \because AB = BA$$

So $\vec{A}\times\vec{B}$ and $\vec{B}\times\vec{A}$ have same magnitude, but direction is opposite

$$\boxed{(\vec{A}\times\vec{B}) = -(\vec{B}\times\vec{A})}$$




② Vector product obeys distributive law

for three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$

$$\boxed{\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}}$$

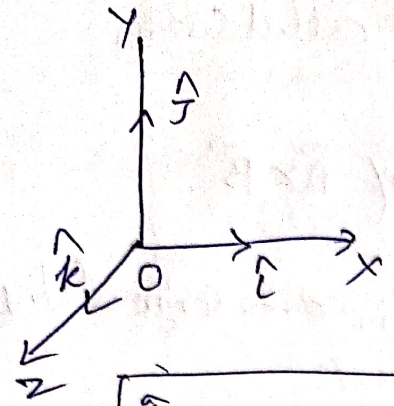
③ For two collinear (Parallel or antiparallel) vectors $\vec{A}$ and $\vec{B}$, $\theta = 0°$ or $180°$ therefore $\sin\theta = 0$

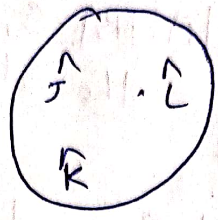
$$\vec{A}\times\vec{B} = AB\sin\theta = 0$$

which gives $\vec{A}\times\vec{B} = 0$ if $\vec{A}$ and $\vec{B}$ are collinear

$\vec{A}\times\vec{A} = 0$ for any vector

(4) For orthonormal unit vectors (for unit vectors of Cartesian Co-ordinate System)




$$\boxed{\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0}$$

$\hat{i}\times\hat{j} = \hat{k}$ and $\hat{j}\times\hat{i} = -\hat{k}$

$\hat{j}\times\hat{k} = \hat{i}$ and $\hat{k}\times\hat{j} = -\hat{i}$

$\hat{k}\times\hat{i} = \hat{j}$ and $\hat{i}\times\hat{k} = -\hat{j}$

(5) Vector Product in term of Cartesian Components

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$$\vec{A}\times\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & \\ B_x & \end{vmatrix}$$

Physical significance of Vector product

Few examples of applications of Vector Product are

(a) Torque $\vec{\tau} = \vec{r}\times\vec{F}$

(b) Angular momentum $\vec{L} = \vec{r}\times\vec{P}$

(c) Angular velocity $\vec{v} = \vec{\omega}\times\vec{r}$

(d) Magnetic force $\vec{F} = q(\vec{v}\times\vec{B})$

(e) Area of Parallelogram Area $= |\vec{A}\times\vec{B}| = AB\sin\theta\,\hat{n}$

## Triple Product .

multiplication of three non zero vectors are known as triple product . There are two types of triple product.

1. Scalar triple product $\vec{A}\cdot(\vec{B}\times\vec{C})$ which gives scalar, so called scalar triple product. it is also denoted $[\vec{A}\vec{B}\vec{C}]$. Scalar triple product of three vectors represent the volume of a parallelopiped, whose three intersecting edges are represented by these vectors

$$\boxed{V = \vec{A}\cdot(\vec{B}\times\vec{C})}$$

- for coplanar vector $\vec{A}$, $\vec{B}$ and $\vec{C}$ $\rightarrow$ $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$

- Scalar triple product in terms of components of vectors -

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$
$$\vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$$

$$\therefore \vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = (B_yC_z - B_zC_y)\hat{i} - (B_xC_z - B_zC_x)\hat{j} + (B_xC_y - B_yC_x)\hat{k}$$

$$\therefore \vec{A}\cdot(\vec{B}\times\vec{C}) = A_x(B_yC_z - B_zC_y) - A_y(B_xC_z - B_zC_x) + A_z(B_xC_y - B_yC_x)$$

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$ $\rightarrow$ which shows scalar triple product as determinant of rectangular components of vectors.

- Scalar triple product of unit vectors of right handed cartesian coordinate system $[\hat{i}\hat{j}\hat{k}] = \hat{i}\cdot(\hat{j}\times\hat{k}) = 1$

Similarly $[\hat{j}\hat{k}\hat{i}] = [\hat{k}\hat{i}\hat{j}] = 1$

and $[\hat{i}\hat{k}\hat{j}] = [\hat{k}\hat{j}\hat{i}] = [\hat{j}\hat{i}\hat{k}] = -1$

→ If any two vectors in scalar triple product are equal their scalar triple product is zero

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

if two vector are equal, then $[\vec{A}\,\vec{A}\,C] = [\vec{C}\,\vec{A}\,\vec{C}] = [\vec{B}\,\vec{B}\,\vec{A}] = 0$

→ If any two vectors of scalar triple product are
→ Colinear, scalar triple product will be zero.

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{A}\cdot(k\vec{A}\times\vec{C}) = k(0) = 0$$

→ Condition for three vectors $\vec{A}, \vec{B}, \vec{C}$ to be Coplanar is $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$

Application of Scalar triple product in physics

→ The scalar triple product finds an important application in Condensed matter physics in the Construction of reciprocal crystal lattice

If $\vec{A}$, $\vec{B}$ and $\vec{C}$ represent the vectors that define a crystal lattice, vectors defining the reciprocal lattice are given as

$$\vec{A}^* = \frac{\vec{B}\times\vec{C}}{\vec{A}\cdot(\vec{B}\times\vec{C})} \qquad \vec{B}^* = \frac{\vec{C}\times\vec{A}}{\vec{A}\cdot(\vec{B}\times\vec{C})} \quad \text{and} \quad \vec{C}^* = \frac{\vec{A}\times\vec{B}}{\vec{A}\cdot(\vec{B}\times\vec{C})}$$

→ In mechanics, Torque of a force about a point O is given as $\vec{\tau} = \vec{r}\times\vec{F}$.

The magnitude of torque of force $\vec{F}$ about a line passing through O is given as

$$|\vec{\tau}_{line}| = \hat{n}\cdot(\vec{r}\times\vec{F})$$

## Vector triple Product :-

If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are three vectors, the quantity $\vec{A}\times(\vec{B}\times\vec{C})$ which gives a vector quantity is called vector triple product of $\vec{A}$ $\vec{B}$ and $\vec{C}$

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

or
$$\vec{A}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} \vec{B} & \vec{C} \\ \vec{A}\cdot\vec{B} & \vec{A}\cdot\vec{C} \end{vmatrix}$$

## Properties of Vector triple product :-

→ Associative law does not hold for vector triple product

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} \quad (i)$$

Now
$$(\vec{A}\times\vec{B})\times\vec{C} = -\vec{C}\times(\vec{A}\cdot\vec{B})$$
$$= -[(\vec{C}\cdot\vec{B})\vec{A} - (\vec{C}\cdot\vec{A})\vec{B}]$$
$$\Rightarrow (\vec{A}\times\vec{B})\times\vec{C} = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{B}\cdot\vec{C})\vec{A} \quad (II)$$

from I and II
$$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$$

→ Jacobi identity -

$$\vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}(\vec{C}\times\vec{A}) + \vec{C}(\vec{A}\times\vec{B}) = 0$$

Proof
$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} \quad (i)$$
$$\vec{B}\times(\vec{C}\times\vec{A}) = (\vec{B}\cdot\vec{A})\vec{C} - (\vec{B}\cdot\vec{C})\vec{A} \quad (ii)$$
$$\vec{C}\times(\vec{A}\times\vec{B}) = (\vec{C}\cdot\vec{B})\vec{A} - (\vec{C}\cdot\vec{A})\vec{B} \quad (iii)$$

Adding eq (i) (ii) and (iii) the identity is proved

Application of vector triple product

① Angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

$$\boxed{\vec{L} = \vec{r} \times m(\vec{\omega} \times \vec{r})}$$

② Centripetal acceleration →

$$\vec{a} = \omega(\omega \times r) \qquad \left(a = \frac{v^2}{r} = \frac{v \cdot v}{r}\right)$$