Vector Calculus

Differentiation of Vectors →

If a vector $\vec{r}$ varies corresponding to the variation of a single scalar variable t in any interval, then $\vec{r}$ is said to be a vector function of scalar variable t and expressed as $\vec{r} = \vec{f}(t)$. Let $\delta\vec{r}$ be the increase in $\vec{r}$ corresponding to the increase δt in the scalar t, Then derivative of Vector function $\vec{r}(t)$ with respect to scalar t is defined as,

$$\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{r}(t+\delta t) - \vec{r}(t)}{\delta t}$$, Provided the limit exists.

when this limit exists, the function $\vec{r}$ is said to be differentiable and the process of determining the derivative of the function is called differentiation.

→ As the vectorial properties of a vector are not changed by its division with a scalar, so the derivative of a vector with respect to a scalar is itself a vector.

→ If $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are vector function of scalar variable t and c is constant vector, then

① $\frac{dc}{dt} = 0$

② $\frac{d}{dt}(\vec{r}_1 + \vec{r}_2) = \frac{d\vec{r}_1}{dt} + \frac{d\vec{r}_2}{dt}$

③ $\frac{d(s\vec{r})}{dt} = \frac{ds}{dt}\vec{r} + s\frac{d\vec{r}}{dt}$

④ If $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ is the Position vector, then
$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$

⑤ $\frac{d(\vec{r}_1 \cdot \vec{r}_2)}{dt} = \frac{d\vec{r}_1}{dt} \cdot \vec{r}_2 + \frac{d\vec{r}_2}{dt} \cdot \vec{r}_1$

If $\vec{r}_1 = \vec{r}_2$ then, $\frac{d}{dt} r^2 = \frac{d}{dt}(\vec{r} \cdot \vec{r}) = 2\vec{r} \cdot \frac{d\vec{r}}{dt}$

⑥ $\frac{d}{dt}(\vec{r}_1 \times \vec{r}_2) = \frac{d\vec{r}_1}{dt} \times \vec{r}_2 + \vec{r}_1 \frac{d\vec{r}_2}{dt}$

⑦ $\frac{d}{dt}[\vec{r}_1 \vec{r}_2 \vec{r}_3] = \frac{d\vec{r}_1}{dt} \cdot (\vec{r}_2 \times \vec{r}_3) + \vec{r}_1 \cdot \left(\frac{d\vec{r}_2}{dt} \times \vec{r}_3\right) + \vec{r}_1 \cdot \left(\vec{r}_2 \times \frac{d\vec{r}_3}{dt}\right)$

$= \left[\frac{d\vec{r}_1}{dt}\, \vec{r}_2\, \vec{r}_3\right] + \left[\vec{r}_1\, \frac{d\vec{r}_2}{dt}\, \vec{r}_3\right] + \left[\vec{r}_1\, \vec{r}_2\, \frac{d\vec{r}_3}{dt}\right]$

⑧ $\frac{d}{dt}[\vec{r}_1 \times (\vec{r}_2 \times \vec{r}_3)] = \frac{d\vec{r}_1}{dt} \times (\vec{r}_2 \times \vec{r}_3) + \vec{r}_1 \times \left(\frac{d\vec{r}_2}{dt} \times \vec{r}_3\right) + \vec{r}_1 \times \left(\vec{r}_2 \times \frac{d\vec{r}_3}{dt}\right)$

Vector differential operator ($\vec{\nabla}$)

The vector differential operator is know as Del operator and denoted by $\vec{\nabla}$. It is vector. It defined as

$$\vec{\nabla} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]$$

$\vec{\nabla}$ operator, operates on vector or scalar as

1 - Gradiant. The gradiant of scalar point function is vector quantity. It defined as $\vec{\nabla}\phi$ or grad ϕ.

$$\text{Grad } \phi = \vec{\nabla}\phi = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\phi = \left[\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right]$$

Properties → grad $(\phi_1 + \phi_2)$ = grad ϕ_1 + grad ϕ_2

grad $(\phi_1 \phi_2)$ = ϕ_1 grad ϕ_2 + ϕ_2 grad ϕ_1

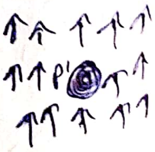
2. Divergen → The divergen of a vector function is scalar quantity. It define as $\vec{\nabla}\cdot\vec{A}$ or div $\vec{A}$.

$$\text{div } \vec{A} = \vec{\nabla}\cdot\vec{A} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\cdot\left[\hat{i}A_x + \hat{j}A_y + \hat{k}A_z\right]$$

$$= \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right]$$

→ divergen of a vector function $\vec{A}$, is a measure of vector How much it spread out, that is diverges from the point (A>0)

→ If the divergence of any vector function in a region is zero (div $\vec{A}$ = 0), this means that the flux of the vector function entering any element of this region is equal to that leaving it. Also div $\vec{A}$ = 0 is called Solenoid Vector



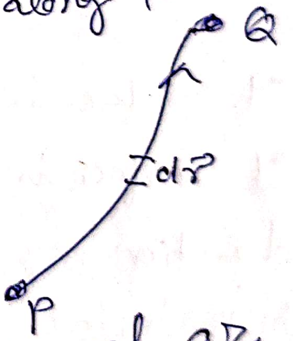
→ The divergence of the vector field at a point is defined as the outward flux of the vector field per unit volume enclosed, through a infinitesimal closed surface surrounding that

(3) **The Curl** - The Curl of a vector field at any point is defined as a vector quantity, whose magnitude is equal to the maximum line integral per unit area along the boundary of an infinitesimal test area at that point and whose direction is perpendicular to the plane of test area. It is also called Circulation or rotation of a vector field. Curl $\vec{A}$ is denoted as $\vec{\nabla} \times \vec{A}$, and given as.

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{i}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] - \hat{j}\left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right] + \hat{k}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

Vector Integration -

Line Integral - The Integral of a point function along a curve is called line integral. It is the integral of the tangential component of the vector along that curve.



Let $\vec{r} = \vec{r}(t)$ is the equation of curve.

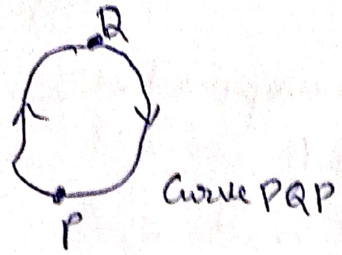
If ϕ and $\vec{A}$ are scalar and vector field respectively and $d\vec{r}$ is the vector increment of length along the curve then possible integral are

$\int_{PQ} \phi \, d\vec{r} \longrightarrow$ vector quantity

$\int_{PQ} \vec{A} \cdot d\vec{r} \longrightarrow$ Scalar "

$\int_{PQ} \vec{A} \times d\vec{r} \longrightarrow$ Vector "

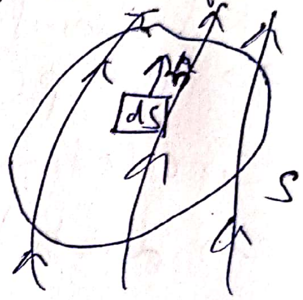
Each of these integral being line integrals along the curve PQ. The curve PQ may be open or closed.



→ If the path of integration is a closed curve as shown in fig, then the line integral become the closed contour integral, that is $\oint \vec{A}\cdot d\vec{r}$

This is also called the circulation of $\vec{A}$ around closed path PQP

Surface Integral – Let S is any surface divided into infinitesimal elements each of which may be considered as vector $d\vec{S}$. Then if ϕ and $\vec{A}$ are scalar and vector fields respectively. Then the surface integral may be written as.



$\iint_S \phi \, d\vec{S} \longrightarrow$ Vector quantity

$\iint_S \vec{A}\cdot d\vec{S} \longrightarrow$ Scalar "

$\iint_S \vec{A} \times d\vec{S} \longrightarrow$ Vector "

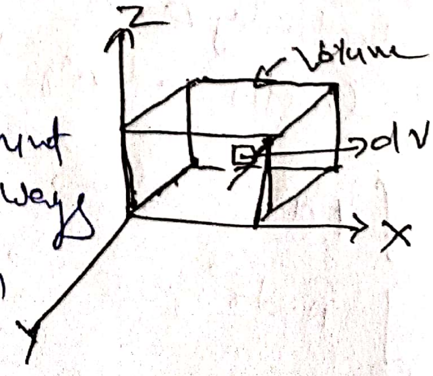
Here area element $d\vec{S}$ may be written as $\hat{n}\,dS$ where $\hat{n}$ is a unit normal vector to indicate the positive direction of the surface

→ If surface is closed then outward normal taken as positive

→ If surface is open then the positive normal depends on the direction in which the perimeter of the open surface is transverse.

→ In particular the surface integral $\iint_S \vec{A}\cdot d\vec{S}$ or flux of the vector function is called the flow.

Volume Integral Let $dV = dx\,dy\,dz$ denotes the element of volume. Since volume element dV is scalar quantity, there only two possible ways in which volume integrals could be written



$\iiint_V \phi \, dV \rightarrow$ Scalar quantity

$\iiint_V \vec{A} \, dV \rightarrow$ Vector "

Gauss divergence theorem. This theorem state that the surface integral of a vector ($\vec{A}$) over a closed surface S equal to volume integral of the divergence of vector over the volume V inclosed by the surface.

$$\iint_S \vec{A}\cdot d\vec{S} = \iiint_V (\vec{\nabla}\cdot\vec{A})\, dV$$

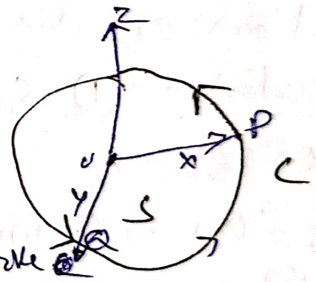
So this theorem converts a volume integral into surface integral and surface to volume also.

Stoke-curl theorem → This theorem state that surface integral of curl of a vector $\vec{A}$ over an open surface S is equal to line integral of vector $\vec{A}$ along around the curve C binding the surface as perimeter.

$$\iint_S \text{curl}\,\vec{A}\cdot d\vec{S} = \oint_C \vec{A}\cdot d\vec{r}$$

or $$\iint (\vec{\nabla}\times\vec{A})\cdot d\vec{S} = \oint_C \vec{A}\cdot d\vec{r}$$

So Stoke theorem converts a surface integral into line integral and line to surface also.



Green's theorem - This theorem state that, if S is a closed region in x-y plane bounded by a simple close curve C and P and Q are continuous function of x and y having continuous derivatives in region then.

$$\oint_C (P\,dx + Q\,dy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where line integral is counter clockwise around the boundary of areas. This is called Green's theorem in the plane.

Helmholtz's theorem, state that

"A vector $\vec{F}$ satisfying $\vec{\nabla}\cdot\vec{F} = S$ (source density or charge density)
$\vec{\nabla}\times\vec{F} = C$ (circulation or current density) with both source and circulation densities vanishing at infinity may be expressed as sum of two parts one of which is irrotational, the other part is solenoidal.

$$\vec{F} = -\vec{\nabla}\phi + \vec{\nabla}\times\vec{A}$$

irrotational solenoidal

Dirac Delta function - is represented by Greek lowercase symbol delta, written as function $\delta(x)$. It is defined by following property,

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \quad \text{--- (1)}$$

with

$$\int_{-\infty}^{\infty} \delta x \, dx = 1 \quad \text{--- (2)}$$

In mathematical literature it is called generalised function or distribution.

Properties - (1) $\delta(-x) = \delta(x)$ that means $\delta(x)$ is even function.

(2) $x\,\delta(x) = 0$

(3) if $f(x)$ is continuous function, $f(x)\delta(x) = f(0)\,\delta(x)$

(4) $\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)$

(5) $\delta(ax) = \frac{1}{a}\delta(x)$, a is constant, $a>0$

(6) Shifting Property → $f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$ (1)

$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$ (2)

(7) Differentiation properties (a) $\delta'(-x) = -\delta'(x)$

(b) $x\frac{d}{dx}\delta(x) = -\delta(x)$