Predicate Calculus: Consider sentences

F: Farhan is a B.Tech Student
R: Rohan is a B.Tech Student

$B(x)$: x is a B.Tech Student $x \in \{$Farhan, Rohan$\}$.
(1 place predicate)

Notes
* Here $B(x)$ is statement function which becomes a statement when x takes the values Farhan/Rohan.
* The set to which x belongs is called Universe or Universe of Discourse

Ex $B(x)$: $x^2+1=0$ then $x \in$ set of complex numbers $= U$
$B(x)$: x is a fruit then $x \in U = \{$grapes, Kiwi, oranges etc$\}$

Some sentences have to be rewritten, so that they can be written in terms of statement function

Ex: (1) All humans are mortal $\Leftrightarrow$ For all x, if x is a human then x is mortal

(2) Some men are clever $\Leftrightarrow$ There exists atleast one x such that x is man & x is clever

Now, to specify whether we are talking about all or few objects/things we need Quantifiers (Expression that indicates scope of term to which they are attached)

(1) $(\forall x) \rightarrow$ Universal quantifier (This is for all)
(2) $(\exists x) \rightarrow$ Existential quantifier (This is for some/few/atleast one etc)

Consider (1) All humans are Mortal
Let $H(x)$: x is human $M(x)$: x is Mortal.
Then $(\forall x)(H(x) \rightarrow M(x))$.

(2) Some men are clever Let $M(x)$: x is a Man
$C(x)$: x is clever
Then $(\exists x)(M(x) \wedge C(x))$

Ex If the universe of discourse is the set $\{a, b, c\}$, eliminate the quantifiers in the following formulas

(a) $(x)\, P(x) \equiv (\forall x)\, P(x)$. $\qquad U = \{a, b, c\}$

$\equiv \{ P(a), P(b), P(c) \}$.

(b) $(x)\, R(x) \wedge (x)\, S(x)$

$\equiv \{ R(a) \wedge S(a), R(a) \wedge S(b), R(a) \wedge S(c), R(b) \wedge S(a), R(b) \wedge S(b), R(b) \wedge S(c), R(c) \wedge S(a), R(c) \wedge S(b), R(c) \wedge S(c) \}$.

(c) $(x)(R(x) \wedge S(x))$

$\equiv \{ R(a) \wedge S(a), R(b) \wedge S(b), R(c) \wedge S(c) \}$

(d) $(x) R(x) \wedge (\exists x) S(x)$

$\equiv (\forall x)\, R(x)$ $\quad$ (Some x)$(S(x))$ (Not all x's have to be considered. Take one or at most 2 values of x)

(e) $(x)(P(x) \to Q(x))$

$\equiv \{ P(a) \to Q(a), P(b) \to Q(b), P(c) \to Q(c) \}$.

Ex Find truth values of

(a) $(x)(P(x) \vee Q(x))$, where $P(x): x = 1$, $Q(x): x = 2$ and the universe of discourse is $\{1, 2\}$

Soln For $x = 1$ $\quad P(1) \vee Q(1) = T \vee F \equiv T$

$P(2) \vee Q(2) = F \vee T = T$

$\therefore (x)(P(x) \vee Q(x)) \to T$.

(b) $(x)[P \to Q(x)] \vee R(a)$, where $P: 2 > 1$, $Q(x): x \leq 3$, $R(x): x > 5$ and $a: 5$, with universe being $\{-2, 3, 6\}$

For $x=-2$ $\quad (P \to Q(-2)) \vee R(5) \equiv (T \to T) \vee F \equiv T \vee F \equiv T.$

For $x=3$ $\quad (P \to Q(3)) \vee R(5) \equiv (T \to T) \vee F \equiv T$

For $x=6$ $\quad (P \to Q(6)) \vee R(5) \equiv (T \to F) \vee F \equiv F \vee F \equiv F$

$\therefore (x)(P \to Q(x)) \vee R(a) \to F$

But $(\exists x)(P \to Q(x)) \vee R(a) \to T$ (why?)

(c) $(\exists x)(P(x) \to Q(x)) \wedge T$, where $P(x): x > 2$, $Q(x): x=0$ and T is any tautology, with the universe of discourse as $\{1\}$

$x=1$ $\quad (P(1) \to Q(1)) \wedge T \equiv (F \to F) \wedge T \equiv T \wedge T \equiv T.$

$\therefore (\exists x)(P(x) \to Q(x)) \wedge T \to T.$

Infact $(\forall x)(P(x) \to Q(x)) \wedge T \to T$ (why??)

Note: * All A are B $\iff (\forall x)$ If x is A then x is B.
* Some A are B $\iff (\exists x)(A(x) \wedge B(x))$.

Free and bound variables.

Ex Indicate variables that are free and bound. Also, show the scope of the quantifiers.

(a) $(x)(P(x) \wedge R(x)) \to (x) P(x) \wedge Q(x)$

Here this is a bound variable $\qquad$ Here x is free variable

Scope: $(x)(P(x) \wedge R(x)) \to$ scope of x here is $(P(x) \wedge R(x))$
$(x) P(x) \to$ " " " " " $P(x)$

(b) $(x)(P(x) \wedge (\exists x) Q(x)) \vee ((x) P(x) \to Q(x))$

These are bound variables $\qquad$ free variable

Scope: $(x)(P(x) \wedge (\exists x) Q(x)) \to (P(x) \wedge (\exists x) Q(x))$; $(\exists x) Q(x) \to Q(x)$
$(x) P(x) \to Q(x) \to P(x)$.

Theory of Inference for Predicate Calculus:

* $\sim((\forall x) A(x)) \equiv (\exists x)(\neg A(x))$
* $\sim((\exists x) A(x)) \equiv \forall(x)(\neg A(x))$

Some Equivalences & Implications

1) $(\exists x)(A(x) \vee B(x)) \Leftrightarrow (\exists x) A(x) \vee (\exists x) B(x)$
2) $(x)(A(x) \wedge B(x)) \Leftrightarrow (x) A(x) \wedge (x) B(x)$
3) $\neg(\exists x) A(x) \Leftrightarrow (x) \neg A(x)$
4) $\neg (x) A(x) \Leftrightarrow (\exists x) \neg A(x)$
5) $(x) A(x) \vee (x) B(x) \Rightarrow (x)(A(x) \vee B(x))$
6) $(\exists x)(A(x) \wedge B(x)) \Rightarrow (\exists x) A(x) \wedge (\exists x) B(x)$

Rule US (Universal Specification): From $(x) A(x)$ one can conclude $A(y)$

Rule ES (Existential specification): From $(\exists x) A(x)$ one can conclude $A(y)$, provided y is not free in any given premise and also not free in any prior step of derivation

[Hint: Choose a new variable every time you use ES].

Rule EG (Existential generalization): From $A(x)$ one can conclude $(\exists y) A(y)$

Rule UG (Universal Generalization): From $A(x)$ one can conclude $(y) A(y)$ provided x is not free in any of the given premises and provided that if x is free in a prior step which resulted from use of ES, then no variables introduced by that use of EG appear free in $A(x)$.

Above are rules of generalization & specification which will be used when checking validity of conclusion from premises.

Ex Demonstrate following implications

(a) $\neg((\exists x)\underset{H1}{P(x)} \wedge Q(a)) \Rightarrow (\exists x)P(x) \rightarrow \overset{C}{\neg Q(a)}$

(1) $\neg((\exists x)P(x) \wedge Q(a))$ H1, Rule P

(2) $(x)\neg P(x) \vee \neg Q(a)$ (1), De Morgans law

(3) $\neg((\exists x)P(x)) \vee \neg Q(a)$ (2), Rule T

(4) $(\exists x)P(x) \rightarrow \neg Q(a)$ (3), Rule T //

(b) $(x)(\neg \underset{H1}{P(x)} \rightarrow Q(x)),\ (x)\underset{H2}{\neg Q(x)} \Rightarrow \underset{C}{P(a)}$

(1) $(x)(\neg P(x) \rightarrow Q(x))$ H1, Rule P

(2) $\neg P(a) \rightarrow Q(a)$ (1) US.

(3) $(x)\neg Q(x)$ H2, Rule P

(4) $\neg Q(a)$ (3), ~~Rule~~ US.

(5) $P(a)$ (2),(4), Modus Tollens //

(c) $(x)(P(x) \vee Q(x)),\ (x)\neg P(x) \Rightarrow (\exists x)Q(x)$

(Exactly same as above question)

(d) $\neg(x)\underset{H1}{(P(x) \wedge Q(x))},\ (x)\underset{H2}{P(x)} \Rightarrow \underset{C}{\neg(x)Q(x)}$

(1) $\neg(x)(P(x) \wedge Q(x))$ H1, Rule P

(2) $(\exists x)(\neg P(x) \vee \neg Q(x))$ (1), De Morgans law

(3) $\neg P(y) \vee \neg Q(y)$ (2) ES.

(4) $(x)P(x)$ H2, Rule P

(5) $P(y)$ (4), US.

(6) $\neg Q(y)$ (4),(5), Rule T

(7) $(\exists x)Q(x)$ (6) EG.

(8) $\neg(x)Q(x)$ (7), Rule T //

Ex Check validity of conclusion from given premises

a) $(\exists x)(P(x) \wedge Q(x))$ $\quad$ C: $(x)P(x)$

(1) $(\exists x)(P(x) \wedge Q(x))$ $\quad$ Rule P

(2) $P(y) \wedge Q(y)$ $\quad$ (1) ES

(3) $P(y)$ $\quad$ (2), Rule T

(4) $(\exists x)P(x) \not\equiv (x)P(x)$ $\quad$ (3), EG $\quad$ ∴ Invalid

b) $\underset{H1}{(x)(P(x) \rightarrow Q(x))}, \underset{H2}{\neg Q(a)}$ $\quad$ C: $(x)\neg P(x)$

(1) $\neg Q(a)$ $\quad$ H2, Rule P

(2) $(x)(P(x) \rightarrow Q(x))$ $\quad$ H1, Rule P

(3) $P(a) \rightarrow Q(a)$ $\quad$ (2), US

(4) $\neg P(a)$ $\quad$ (1),(3), Modus Tollens

(5) $(\exists x)\neg P(x) \not\equiv (x)\neg P(x)$ $\quad$ (4) EG $\quad$ Invalid

Ex Using Rule CP or otherwise show following implications:

a) $(x)(P(x) \rightarrow Q(x)) \Rightarrow (x)P(x) \rightarrow (x)Q(x)$

(1) $(x)(P(x) \rightarrow Q(x))$ $\quad$ H1, Rule P

(2) $(x)(\neg P(x) \vee Q(x))$ $\quad$ (1), Rule T

(3) $\neg P(y) \vee Q(y)$ $\quad$ (2) US

(4) $(x)P(x)$ $\quad$ Rule CP

(5) $P(y)$ $\quad$ (4) US

(6) $Q(y)$ $\quad$ (3),(5), Rule T

(7) $(x)Q(x)$ $\quad$ (6), UG

(8) $(x)P(x) \rightarrow (x)Q(x)$ $\quad$ (4),(7), Rule T //

(b) $(\exists x) P(x) \rightarrow (x) Q(x) \Rightarrow (x)(P(x) \rightarrow Q(x))$

(1) $(\exists x) P(x) \rightarrow (x) Q(x)$ Rule P

(2) $\neg(\exists x) P(x) \vee (x) Q(x)$ (1), Rule T

(3) $(x) \neg P(x) \vee (x) Q(x)$ (2), Rule T

(4) $(x)(\neg P(x) \vee Q(x))$ as $(x) A(x) \vee (x) B(x) \Rightarrow (x)(A(x) \vee B(x))$

(5) $(x)(P(x) \rightarrow Q(x))$ (4), Rule T

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[Note Here you cannot apply rule CP as (x) is before $(P(x) \rightarrow Q(x))$ & not $(x) P(x)$]

Ex Show that $P \rightarrow (\exists x) Q(x) \Leftrightarrow (\exists x)(P \rightarrow Q(x))$

Soln $P \rightarrow (\exists x) Q(x)$

$\Leftrightarrow \neg P \vee (\exists x) Q(x)$

$\Leftrightarrow \neg P \vee Q(y)$ ES

$\Rightarrow P \rightarrow Q(y)$ Rule T

$\Rightarrow (\exists x)(P \rightarrow Q(x))$ EG — (1)

Converse

$(\exists x)(P \rightarrow Q(x))$

$\Rightarrow (\exists x)(\neg P \vee Q(x))$

$\Rightarrow \neg P \vee (\exists x) Q(x)$

$\Rightarrow P \rightarrow (\exists x) Q(x)$ — (2)

From (1) & (2) $P \rightarrow (\exists x) Q(x) \Leftrightarrow (\exists x)(P \rightarrow Q(x))$ //