Base: Let (X, T) be a topological space. A sub family B of T is said to be a base for T if every member of T can be expressed as the union of some members of B

OR

Let $B \subset T$ s.t. $B \neq \phi$

B is said to be a base or open base or basis for the topology T on X if

given any non-empty set $G \in T \Rightarrow \exists\ B_1 \subset B$

s.t. $G = \cup \{ B_1 : B_1 \in B \}$.

Another definition.

B is said to be a base for the topology T on X if

$x \in G \in T \Rightarrow \exists\ B_1 \in B$ s.t. $x \in B_1 \subset G$

e.g.

In a metric space every open set can be expressed as a union of open spheres and the family of all open spheres is a base for the topology induced by metric.

The set of all open intervals in R form a base for the usual topology on R.

Sub Base:-

Let (X, T) be a topological space. Let

$S \subset T$ s.t. $S \neq \phi$

S is said to be a sub base or open sub base for the topology T on X if finite intersections of the members of S form a base for T on X i.e the unions of members of S give all the members of T. The element of S are referred to as sub basic open sets.

Example. Let $a, b \in R$ be arbitrary s.t. $a < b$. Evidently

$$(-\infty, b) \cap (a, \infty) = (a, b)$$

The open interval (a, b) form a base for the usual topology on R. Hence by def^n ~~infinite~~ the family of infinite open intervals forms a sub base for the usual topology on R.

Local Base.

Let (X, T) be a topological space. A family B_x of open subsets of X is said to be local base at $x \in X$ for T on X if

(i) $B \in B_x \Rightarrow x \in B$

(ii) any $G \in T$ with $y \in G \Rightarrow B \in B_x$ s.t. $y \in B \subset G$

Example. Let $x \in R$ be arbitrary

write $B_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \ \forall n \in N$

take $B_x = \{B_n : n \in N\}$.

Evidently B_x is a local base at $x \in X$ for the usual topology on R.

First countable space -

Let (X, T) be a topological space. The space X is said to satisfy the first axiom of countability if X has a countable local base at each $x \in X$. The space X, in this case is called first countable space.

In the above example B_x is countable local base under map $B_n \to n$. Hence R with usual topology is first countable.

Second countable space.

Let (X, T) be topological space. The space X is said to satisfy the second axiom of countability if $\exists$ a countable base for T on X.

In this case the space X is called second countable or second axiom space.

A second countable space is also called completely separable.

Example - The set of all open intervals (r, s) with r and s as rational numbers form a base say B for the usual topology U of R. Since Q, $Q \times Q$ are countable sets and so B is countable base for U on R.