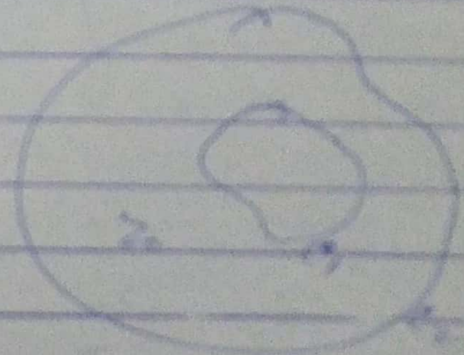
## Extension of Cauchy's Integral Formula -

If $f(z)$ is analytic in a ring shaped region bounded by two closed curves $C_1$ & $C_2$ and ~~...~~ $z_0$ is a point in the region between $C_1$ and $C_2$

$$f(z_0) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}\,dz$$

where $C_2$ is outer curve.




## Cauchy Integral Formula for the derivative of an analytic function -

If a function $f(z)$ is analytic within and on a closed contour $C$ and $a$ is any point lying in it, then

$$f'(a) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^2}\,dz$$

Proof.

Let $a+h$ be point in the nbd of $a$, then

by Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

and $$f(a+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-(a+h)}$$

from which we get

$$\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{h}\left[\frac{1}{z-a-h} - \frac{1}{z-a}\right]dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)h}\left[\left(1-\frac{h}{z-a}\right)^{-1} - 1\right]dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)h}\left[\left(1+\frac{h}{(z-a)}+\left(\frac{h}{z-a}\right)^2+\cdots\right)-1\right]dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}\left[\frac{1}{z-a}+\frac{h}{(z-a)^2}+\frac{h^2}{(z-a)^3}+\cdots\right]$$

Now

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}\left[\frac{1}{z-a}+0+0+\cdots\right]$$

or $$\boxed{f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz}$$

**Higher order derivatives** - If a function $f(z)$ is analytic within and on a closed contour $C$ and $a$ is any point within $C$ then derivative of all orders are analytic and are given by

$$f^{n}(a) = \frac{n!}{2\pi i}\int_{C} \frac{f(z)\,dz}{(z-a)^{n+1}}$$

**Proof**

we know that

$$f'(a) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^2}\,dz \quad \text{--- (1)}$$

This proves that the required result is true for $n = 1$. Let us suppose that the required result is true for $n = m$ so that

$$f^{m}(a) = \frac{m!}{2\pi i}\int_{C} \frac{f(z)\,dz}{(z-a)^{m+1}}$$

Let $a+h$ be a point in the nbd of $a$. Observe that

$$\frac{f^{m}(a+h) - f^{m}(a)}{h} = \frac{m!}{2\pi i h}\int_{C} f(z)\left\{\frac{1}{(z-a-h)^{m+1}} - \frac{1}{(z-a)^{m+1}}\right\} dz$$

$$= \frac{m!}{2\pi i h}\int_{C} \frac{f(z)}{(z-a)^{m+1}}\left\{\left(1-\frac{h}{z-a}\right)^{-(m+1)} - 1\right\} dz$$

$$= \frac{m!}{2\pi i h}\int_{C} \frac{f(z)}{(z-a)^{m+1}}\left\{\frac{(m+1)h}{z-a} + \frac{(m+1)(m+2)}{2!}\left(\frac{h}{z-a}\right)^2 + \cdots\right\} dz$$

$$= \frac{m!}{2\pi i}\int_{C} \frac{f(z)}{(z-a)^{m+1}}\left[\frac{(m+1)}{(z-a)} + \frac{(m+1)(m+2)}{2!}\frac{h}{(z-a)^2} + \cdots\right] dz$$

$$= \frac{(m+1)!}{2\pi i}\int_{C} \frac{f(z)}{(z-a)^{m+1}}\left[\frac{1}{z-a} + \frac{(m+2)}{2!}\frac{h}{(z-a)^2} + \cdots\right] dz$$

or $$f^{m+1}(a) = \frac{(m+1)!}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^{m+2}} \quad \text{(by taking limit as } h \to 0\text{)}$$

## Morera's Theorem

If a function $f$ is continuous throught a domain $D$ and if

$$\oint_C f(z)\,dz = 0$$

for every closed contour $C$ lying in $D$, then $f$ is analytic throughout $D$.

(It is a sort of converse of the Cauchy Theo

**Proof** Let $z_0$ be a fixed point and $z$ a variable point inside the domain $D$.

The value of the integral $\int_{z_0}^{z} f(t)\,dt$ is independent of the curve joining $z_0$ to $z$ and depends on $z$ only.

write $F(z) = \int_{z_0}^{z} f(t)\,dt$

Let $z+h$ be a point nbd of $z$

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(t)\,dt - \int_{z_0}^{z} f(t)\,dt$$

$$= \int_{z_0}^{z+h} f(t)\,dt + \int_{z}^{z_0} f(t)\,dt$$

$$= \int_{z}^{z+h} f(t)\,dt$$

Now

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \cancel{\frac{1}{|h|}\left|\int_{z}^{z+h} \{f(t) - f(z)\}\,dt\right|}$$

$$= \frac{1}{|h|}\left|\int_z^{z+h} f(t)\,dt - f(z)\right|$$

$$= \frac{1}{|h|}\left|\int_z^{z+h} f(t)\,dt - \frac{1}{h}\int_z^{z+h} f(z)\,dt\right|$$

$$= \frac{1}{|h|}\left|\int_z^{z+h}\{f(t)-f(z)\}\,dt\right|$$

$$\boxed{\because \int_z^{z+h} f(z)\,dt = hf(z)}$$

$$\le \frac{1}{|h|}\int_z^{z+h}|f(t)-f(z)|\,|dt| < \frac{\epsilon}{|h|}|h|$$

(Since $f$ is continuous so that $|f(t)-f(z)| < \epsilon$ for $|t-z| < \delta$)

or $$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| < \epsilon$$

which tends to zero as $\epsilon \to 0$

Thus $$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} - f(z) = 0$$

or $$F'(z) = f(z)$$

Thus the derivative of $F(z)$ exists and so $F(z)$ is analytic function in $D$. ~~and~~ also we know that the derivative of analytic function is analytic.

Therefore $f(z)$ is analytic in $D$.

**Cauchy's Inequality** = If $f(z)$ is analytic within and on a circle $C$, given by $|z-a| = R$ and $|f(z)| \le M$ for every $z$ on $C$ then

$$|f^n(a)| \le \frac{M\, n!}{R^n}$$

**Proof** $|z-a| = R \Rightarrow z-a = Re^{i\theta} \Rightarrow dz = iRe^{i\theta}\, d\theta$

$$\Rightarrow |dz| = R\, d\theta$$

we know that

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)\, dz}{(z-a)^{n+1}}$$

$$|f^n(a)| \le \frac{n!}{2\pi i} \int_C \frac{|f(z)| \cdot |dz|}{|z-a|^{n+1}}$$

$$\le \frac{M n!}{2\pi R^{n+1}} \int_0^{2\pi} R\, d\theta$$

$\because |e^{i\theta}| = 1$

$$= \frac{M n!}{2\pi R^{n+1}} 2\pi R$$

$$|f^n(a)| \le \frac{M\, n!}{R^n}$$

**Remark**- If we have $a_n = \frac{f^n(a)}{n!}$ then

$$|a_n| \le \frac{M}{R^n}$$

**Integral function** - A function $f(z)$ is called an integral function or entire function if it is analytic in every finite region.

## Liouville's Theorem → If $f(z)$ is entire and is bounded for all values of $z$ in the complex plane, then $f(z)$ is constant throughout the plane.

OR

If a function $f(z)$ is analytic for finite values of $z$ and is bounded then $f(z)$ is constant.

**Proof.** Let $a$ and $b$ be arbitrary points in $z$-plane and let $C$ be a large circle with centre $z=0$ and radius $R$ such that $C$ encloses $a$ and $b$.

Eqⁿ of $C$ is $|z| = R \Rightarrow z = Re^{i\theta}$

$\Rightarrow dz = iRe^{i\theta} d\theta$

$\Rightarrow |dz| = R\, d\theta$

Let $f(z)$ is bounded $\forall z$

$\Rightarrow |f(z)| \leq M \quad \forall z$

where $M > 0$

By Cauchy's Integral Formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a} \quad \& \quad f(b) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-b}$$

$$f(a) - f(b) = \frac{1}{2\pi i}\int_C \left(\frac{1}{z-a} - \frac{1}{z-b}\right) f(z)\,dz$$

$$= \frac{(a-b)}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)(z-b)}$$

$$|f(a) - f(b)| \leq \frac{|a-b|}{2\pi}\int_C \frac{|f(z)|\,|dz|}{(|z|-|a|)(|z|-|b|)}$$

$$\leq \frac{M\,|a-b|\,2\pi R}{2\pi\,(R-|a|)(R-|b|)}$$

$$|f(a) - f(b)| \leq \frac{MR\,|a-b|}{(R-|a|)(R-|b|)} \to 0 \text{ as } R \to \infty$$

i.e. $f(a) - f(b) = 0$

or $f(a) = f(b)$ showing thereby $f(z)$ is constant.

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