Continuous Mapping: Let X and Y be metric spaces with metrics d_1 and d_2 and let f be a mapping of X into Y. f is said to be continuous at a point x_0 in X if either of following equivalent condition is satisfied

(1) for each $\epsilon > 0$ there exists $\delta > 0$ such that $d_1(x, x_0) < \delta$
$\Rightarrow d_2(f(x), f(x_0)) < \epsilon$

(2) for each open sphere $S_\epsilon(f(x_0))$ centered on $f(x_0)$ ∃ an open sphere $S_\delta(x_0)$ centered at x_0 such that $f(S_\delta(x_0)) \subseteq S_\epsilon(f(x_0))$

Theorem 1: Let X and Y be metric space and f is a mapping of X into Y. Then f is continuous at x_0 iff $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

or f is continuous $\Rightarrow$ f is sequentially continuous

Proof:

Let $f: X \to Y$ be continuous mapping at $x_0 \in X$ and let $\langle x_n \rangle$ be sequence in X s.t. $x_n \to x_0$

Aim – f is sequentially continuous

For this we have to show that
$$f(x_n) \to f(x_0)$$

By continuity of f at x_0

given $\epsilon > 0$ ∃ $\delta > 0$ s.t.
$$d(x, x_0) < \delta \Rightarrow \rho[f(x), f(x_0)] < \epsilon \quad \text{--- (1)}$$

Since $x_n \to x_0$

and so given $\delta > 0$ ∃ $n_0 \in N$ s.t.
$$d(x_n, x_0) < \delta \quad \forall\, n \geq n_0 \quad \text{--- (2)}$$

writing (1) with help of (2) we get

given $\epsilon > 0$ ∃ $\delta > 0$ s.t.
$$d(x_n, x_0) < \delta \Rightarrow \rho[f(x_n), f(x_0)] < \epsilon \quad \forall\, n \geq n_0$$

or $\rho[f(x_n), f(x_0)] < \epsilon \quad \forall\, n \geq n_0$

This $\Rightarrow f(x_n) \to f(x_0)$

Conversely

suppose that
$$x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$$

To prove that f is continuous at x_0.

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Suppose not i.e f is not continuous at x_0 then given $\epsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$d(x_n, x_0) < \delta \Rightarrow \rho(f(x_n), f(x_0)) \nless \epsilon \quad \text{--- (3)}$$

Now consider the sequence of open spheres s.t. $S_1(x_0), S_{1/2}(x_0) \ldots S_{1/n}(x_0)$

Form a sequence $\langle x_n \rangle$ s.t.

$$x_n \in S_{1/n}(x_0)$$

Evidently $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

$\Rightarrow \forall n \geq n_0 \quad \rho[f(x_n), f(x_0)] < \epsilon$

But form (3)

$$\rho[f(x_n), f(x_0)] \nless \epsilon$$

i.e

$$\lim_{n \to \infty} f(x_n) \neq f(x_0)$$

A contradiction. Hence f must be continuous at x_0.

(V. Imp)

Theorem (2) Let X and Y be metric spaces and f a mapping of X into Y. Then f is continuous $\Leftrightarrow$ $f^{-1}(G)$ is open in X whenever G is open in Y.

Proof - we first assume that f is continuous. If G is an open set in Y we must show that $f^{-1}(G)$ is open in X.

$f^{-1}(G)$ is open if it is empty

Let $f^{-1}(G) \neq \phi$. Let $x \in f^{-1}(G)$. Then $f(x)$ is in G and since G is open, $\exists$ an open sphere $S_\epsilon(f(x))$ centred at $f(x)$ and contained in G.

By def. of continuity $\exists$ an open sphere $S_\delta(x)$ s.t. $f(S_\delta(x)) \subseteq S_\epsilon(f(x))$ since $S_\epsilon(f(x)) \subseteq G$ we also have $f(S_\delta(x)) \subseteq G$ and from this we see that $S_\delta(x) \subseteq f^{-1}(G)$. $S_\delta(x)$ is an open sphere centred on x and contained in $f^{-1}(G)$ so $f^{-1}(G)$ is open.

we now assume that $f^{-1}(G)$ is open in X whenever G is open in Y.

To prove that f is continuous

We show that f is continuous at an arbitrary point x in X. Let $S_\epsilon(f(x))$ be an open sphere; this open sphere is an open set, so its inverse image is an open set in X which contains x. By this $\exists$ an open sphere $S_\delta(x)$ in X contained in this inverse image. It is clear that $f(S_\delta(x))$ is contained in $S_\epsilon(f(x))$, so f is continuous at x. Finally since x was taken to be arbitrary point in X, f is continuous.

Uniform continuity —

Uniform continuity is essentially continuity plus the added condition that for each ϵ we can find a δ which works uniformly over the entire space X, in the sense that it does not depend on x. The formal def. is as follows – If X and Y are metric spaces with d_1 and d_2 metrices, then a mapping f of X into Y is said to be uniformly continuous if for each $\epsilon > 0$ there exists $\delta > 0$ such that $d_1(x, x') < \delta \Rightarrow d_2[f(x), f(x')] < \epsilon$. It is clear that any uniformly continuous mapping is automatically continuous.

For example:

The real function defined by $f(x) = 2x$ is uniformly continuous. On the other hand the real function g defined on R by $g(x) = x^2$ is continuous but not uniform continuous.

Uniformly continuous continuity is property of function on an entire set where as continuity can be defined on a single point. Uniform continuity on a point has no sense.

"A map $f : (X, d) \to (Y, \rho)$ is called isomorphism iff f is bijective and both f and f^{-1} is uniformly continuous."

Extension Theorem :- Let (X, d) and (Y, ρ) be metric spaces and Y is complete. Let $A \subset X$ be dense. If $f: A \to Y$ be uniformly continuous then f can be extended uniquely to a uniform continuous map $g: X \to Y$.

Proof: Let $A \subset X$ be dense so that $\bar{A} = X$ — ①

Also let $f: A \to (Y, \rho)$ be uniformly continuous map so that given $\epsilon > 0$ $\exists$ $\delta > 0$ s.t. $a, b \in A$ and

$$d(a, b) < \delta \Rightarrow \rho[f(a), f(b)] < \epsilon$$ ②

And (Y, ρ) be complete so that every cauchy sequence in ~~convergent~~ Y converges to some point in Y. —— ③

To prove that f can be extended to a uniform continuous map $g: X \to Y$

we define $g: X \to Y$ s.t. $g(x) = f(x)$ $\forall x \in X$

If $A = X$ then there is nothing to prove ④

So let $A \neq X$ since $\bar{A} = A \cup D(A)$. In this event ① proves that

$$X - A = D(A)$$ ⑤

Let $x \in X - A$ be arbitrary. For this x, we define $g(x)$ as follows

Since $x \in X - A$

$\Rightarrow x \in D(A)$

$\Rightarrow x$ is limit point of A

$\Rightarrow \exists$ a sequence $\langle a_n \rangle$ in A s.t. $a_n \to x$

$\Rightarrow a_n$ is cauchy sequence s.t. $\lim a_n = x$

[for every convergent sequence is cauchy sequence]

$\Rightarrow f(a_n)$ is cauchy sequence in Y

(For the image of ~~continuous~~ cauchy sequence under uniform continuous map is cauchy sequence since Y is complete.

In order to show that g is well defined, we have to show that g depends only on x

Let $\langle b_n \rangle$ be another sequence in A s.t. $b_n \to x$

Then $d(a_n, b_n) \to 0$ so that $\rho[f(a_n), f(b_n)] \to 0$
[For f is uniform continuous]

But $f(a_n) \to g(x)$ so that $f(b_n) \to g(x)$

Hence $g(x)$ depends on x only

To show that g is uniformly continuous

Let $x, y \in X$ be arbitrary then either $x, y \in A$ or x, y are limit point of A according to (1)

In every case $\exists$ sequence $\langle x_n \rangle$, $\langle y_n \rangle$ in A s.t. $x_n \to x$, $y_n \to y$

This $\Rightarrow$ $d(x_n, x) \to 0$ $d(y_n, y) \to 0$ — (6)

By def. of g $f(x_n) = g(x)$ & $f(y_n) = g(y)$

This $\Rightarrow$ $\rho[f(x_n) f(y_n)] \to \rho[g(x), g(y)]$ (7)
— (8)

But ~~$d(x_n, y)$~~ $d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$

making using (6), (7), (8) and taking limit as $n \to \infty$

we get $d(x_n, y_n) \leq d(x, y) < \delta$
or $d(x_n, y_n) < \delta$

So that $\rho[f(x_n), f(y_n)] < \epsilon$ for large n using (8)

$\rho[g(x), g(y)] < \epsilon$

Thus $d(x, y) < \delta \Rightarrow \rho[g(x), g(y)] < \epsilon$

$\therefore$ g is uniformly continuous

To prove that g is unique

Let g' be another extension of f

Then $g'(x) = g(x)$ $\forall$ $x \in A$

Also g and g' are continuous

[for uniform continuity on entire

This $\Rightarrow g'(x) = g(x) \ \forall x \in \bar{A}$

using (1)

$$g'(x) = g(x) \quad \forall x \in X$$

$\therefore$ g is unique

Theorem: Let $X(d)$ and (Y, ρ) be metric spaces. Let $A \subset X$ & f, g are continuous maps from X into Y s.t.

$$f(x) = g(x) \quad \forall x \in A$$

then

$$f(x) = g(x) \quad \forall x \in \bar{A}$$

PROOF: Since $A \subset \bar{A}$. Hence if we show that

$$f(x) = g(x) \quad \forall x \in \bar{A} - A$$

the result will be proved

Let $x \in \bar{A} - A$

$\Rightarrow x \in \bar{A}$

$\Rightarrow x \notin A$

$\Rightarrow x \in D(A) \qquad \because \bar{A} = A \cup D(A)$

$\Rightarrow x$ is limit point of A s.t. $x \notin A$

x is limit point of A

$\Rightarrow \exists$ a sequence $\langle x_n \rangle$ in A s.t. $x_n \to x$

Also f and g are continuous at x

Hence $f(x_n) \to f(x)$

$g(x_n) \to g(x)$

$x_n \in A \ \forall n \Rightarrow f(x_n) = g(x_n) \ \forall n$ by assumption

$= \lim f(x_n) = \lim g(x_n)$

$\Rightarrow f(x) = g(x)$

$\therefore f(x) = g(x)$ with $x \in \bar{A} - A$

finally $f(x) = g(x) \quad \forall x \in \bar{A}$ Proved