Caley Theorem :- (and its applications)

Theorem :- Every group is isomorphic to a group of permutations.

Proof :- Let G be any group. we must find a group $\overline{G}$ of permutations that we believe is isomorphic to G. Since G is all we have to work with, we will have to use it to construct $\overline{G}$.

now for any $g \in G$, define a function

- $T_g : G \longrightarrow G$

  $T_g(x) = gx$, for all x in G

- $$\begin{aligned} T_g T_h(x) &= T_g(T_h(x)) \\ &= T_g(hx) \\ &= ghx \\ &= (gh)x \\ &= T_{gh}(x) \end{aligned}$$

so $\boxed{T_g T_h = T_{gh}}$

(1)

Example:- $G = U(12) = \{1, 5, 7, 11\}$

$g \in G = U(12)$

$g = 1$, $T_g(x) = T_1(x) = 1x = x$

$T_1(1) = 1$
$T_1(5) = 5$
$T_1(7) = 7$
$T_1(11) = 11$

$T_1 = \begin{pmatrix} 1 & 5 & 7 & 11 \\ 1 & 5 & 7 & 11 \end{pmatrix}$

$g = 5$, $x \in G$ $T_5(x) = 5x \Rightarrow T_5(1) = 5$
$T_5(5) = 25 \bmod 12 = 1$
$T_5(7) = 35 \bmod 12 = 11$
$T_5(11) = 55 \bmod 12 = 7$

$T_5 = \begin{pmatrix} 1 & 5 & 7 & 11 \\ 5 & 1 & 11 & 7 \end{pmatrix}$

$g = 7$, $x \in G$ $T_7(x) = 7x \Rightarrow T_7(1) = 7$
$T_7(5) = 35 \bmod 12 = 11$
$T_7(7) = 49 \bmod 12 = 1$
$T_7(11) = 77 \bmod 12 = 5$

$T_7 = \begin{pmatrix} 1 & 5 & 7 & 11 \\ 7 & 11 & 1 & 5 \end{pmatrix}$

$g = 11$, $x \in G$ $T_{11}(x) = 11x \Rightarrow T_{11}(1) = 11$
$T_{11}(5) = 55 \bmod 12 = 7$
$T_{11}(7) = 77 \bmod 12 = 5$
$T_{11}(11) = 121 \bmod 12 = 1$

$T_{11} = \begin{pmatrix} 1 & 5 & 7 & 11 \\ 11 & 7 & 5 & 1 \end{pmatrix}$

$\overline{G} = \overline{U(12)} = \{T_1, T_5, T_7, T_{11}\}$

$U(12)$

$\cdot_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

$(\overline{U(12)})$

$\cdot_{12}$	T_1	T_5	T_7	T_{11}
T_1	T_1	T_5	T_7	T_{11}
T_5	T_5	T_1	T_{11}	T_7
T_7	T_7	T_{11}	T_1	T_5
T_{11}	T_{11}	T_7	T_5	T_1

$U(12) \cong \overline{U(12)}$

Ques ② $G = \mathbb{Z}_5$ $(\mathbb{Z}_5, +)$ is group.

Show that $(\mathbb{Z}_5, +)$ is isomorphic to its permutation group

$$T_g(x) = g(x) \quad x \in G$$

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$

$\bar{G} = \overline{\mathbb{Z}_5} = \{T_0, T_1, T_2, T_3, T_4\}$