Characteristic of Topology in terms of Kuratowski closures operator

Kuratowski closure operator – Let (X, T) be topological space. A closure operator on X is a function $c : P(X) \rightarrow P(X)$ satisfying the following four conditions known as **Kuratowski closure axioms**:

[K1] $c(\phi) = \phi$

[K2] $A \subset c(A)$

[K3] $c(A \cup B) = c(A) \cup c(B)$

[K4] $c(c(A)) = c(A)$

where A and B are any subset of X.

It is found that there are many different ways of defining a topology other than open set approach e.g. closed set approach, nbd approach, closure approach etc.

Here we associate a topology T on X with a closure operator c in such a way that T-closure of each subset A of X is precisely $c(A)$.

Theorem – Let X be any set and c be the Kuratowski closure operator on X then there exists a unique topology T on X s.t. for each $A \subset X$, $c(A)$ coincides with T-closure of A.

Proof :- Let c be the Kuratowski closure operator on X i.e. a function

$$c : P(X) \to P(X) \text{ such that}$$

[K1] $c(\phi) = \phi$

[K2] $A \subset c(A) \quad \forall A \subset X$

[K3] $c(A \cup B) = c(A) \cup c(B) \quad \forall A, B \subset X$

[K4] $c(c(A)) = c(A) \quad \forall A \subset X$

Let $\mathcal{F}$ be the collection of all subsets F of X such that $c(F) = F$ i.e.

$$\mathcal{F} = \{F : F \subset X \text{ and } c(F) = F\}$$

We first show that $\mathcal{F}$ satisfies all the following conditions of characterization of a topological space in terms of the closed set

(i) $\phi \in \mathcal{F}, X \in \mathcal{F}$

(ii) $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cup F_2 \in \mathcal{F}$

(iii) $F_\lambda \in \mathcal{F} \ \forall \lambda \in \Lambda \Rightarrow \cap\{F_\lambda : \lambda \in \Lambda\} \in \mathcal{F}$

(i) By [K1], $\phi \in \mathcal{F}$

By [K2] $X \subset c(X)$ but $c(X) \subset X$ always

so $c(X) = X \Rightarrow X \in \mathcal{F}$

(ii) Let $F_1, F_2 \in \mathcal{F} \Rightarrow c(F_1) = F_1$ & $c(F_2) = F_2$

$\Rightarrow c(F_1) \cup c(F_2) = F_1 \cup F_2$

$\Rightarrow c(F_1 \cup F_2) = F_1 \cup F_2$ by [K2]

$\Rightarrow F_1 \cup F_2 \in \mathcal{F}$ ~~[illegible]~~

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