Exponential function

$$f(z) = e^{z} = e^{x+iy} = e^{x} e^{iy}$$
$$= e^{x}(\cos y + i \sin y)$$
$$= e^{x}\cos y + i e^{x}\sin y$$
$$= u + iv.$$

Trigonometric functions.

$$f(z) = \sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$$
$$= \sin x \cosh y + i \cos x \sinh y$$
$$= u + iV$$

$$f(z) = \cos z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy$$
$$= \cos x \cosh y - i \sin x \sinh y$$
$$= (u+iv)$$

etc.

Hyperbolic functions.

$$f(z) = \sinh z = -i \sin iz = -i \sin(ix - y)$$
$$= -i[\sin ix \cos y - \cos ix \sin y]$$
$$= -i[i \sinh x \cos y - \cosh x \sin y]$$
$$= \sinh x \cos y + i \cosh x \sin y$$
$$= (u+iv)$$

etc.

Logarithmic function.

$$f(z) = \log z = \log(x+iy)$$
$$= \frac{1}{2}\log(x^2+y^2) + i \tan^{-1}(y/x) = u+iv$$

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Ex- Show that function $f(z) = \sqrt{|xy|}$ is not analytic at origin although the C.R. equations are satisfied at that point

Soln.
$$f(z) = \sqrt{|xy|}$$

then $u = \sqrt{|xy|}$, $v = 0$

At the origin we have

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

And $\because v = 0 \Rightarrow \partial v/\partial x = 0$ & $\partial v/\partial y = 0$

Hence C.R. eqns are satisfied at origin

Now
$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z\to 0} \frac{\sqrt{|xy|} - 0}{x + iy} \quad \text{--- (1)}$$

Suppose $z \to 0$ along path $y = mx$ then we have

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|mx^2|} - 0}{(1+im)x}$$

$$= \lim_{x\to 0} \frac{\sqrt{|m|}}{1+im} = \frac{\sqrt{|m|}}{1+im}$$

which depends on m so it is not unique

$\therefore f'(0)$ does not exists

Therefore $f(z)$ is not analytic at origin.

CONSTRUCTION OF ANALYTIC FUNCTIONS

① Milne's Thomson method

we have to discussed the formation of the analytic function under the following cases

(I) when real part i.e. u is given
(II) when imaginary part i.e. v is given
(III) when $u-v$ is given
(IV) when $u+v$ is given.

Case-I :- when u is given.

working:

(i) Find $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$

(ii) write $\phi_1(z,0) = \left(\frac{\partial u}{\partial x}\right)_{(z,0)}$ & $\phi_2(z,0) = \left(\frac{\partial u}{\partial y}\right)_{(z,0)}$

(iii) Then by Milne's Thomson Method

$$f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + c$$

Case-II :- when v is given

(i) find $\partial v/\partial x$ & $\partial v/\partial y$

(ii) Write $\psi_1(z,0) = \left(\frac{\partial v}{\partial x}\right)_{(z,0)}$ & $\psi_2(z,0) = \left(\frac{\partial v}{\partial y}\right)_{(z,0)}$

(iii) then by Milne's Thomson method

$$f(z) = \int [\psi_2(z,0) + i\,\psi_1(z,0)]\,dz + c$$

Case-III :- when $(u-v)$ is given.

let $f(z) = u+iv$

so $if(z) = iu - v$

Adding

$$(1+i)f(z) = (u-v) + i(u+v)$$

i.e. $F(z) = U+iV$ —— ①

where $F(z) = (1+i)f(z)$ $U = u-v$ & $V = u+v$

$(u-v)$ is given it means real part i.e. U of $F(z)$ be given so that we first find $F(z)$ by proceeding as in case I then the required function

$$f(z) = \frac{F(z)}{(1+i)}$$

<u>Case IV</u> <u>when $(u+v)$ is given</u>

we have $f(z) = u+iv$

$if(z) = iu - v$

Adding

$$(1+i)f(z) = (u-v) + i(u+v)$$

$$\Rightarrow F(z) = U+iV$$

where $V = u+v$

so $(u+v)$ is given means V is given and in this case first we find $F(z)$ by proceeding as in case II and then the required function

$$f(z) = \frac{F(z)}{(1+i)}$$