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Illustrations :—

(1) The entire set $R = (-\infty, \infty)$ is open if $x \in R$, then R is a nbd of x.

(2) Any open interval (a, b) is an open set for if $x \in (a, b)$, then (a, b) is a nbd of x, and thus x is an interior point of (a, b).

(3) The set $[a, b]$ is not open, since every nbd of a and that of b contains points not in $[a, b]$.

(4) The sets $[a, b)$ and $(a, b]$ are also not open.
(5) The null set ϕ is open in R.

17/08 Countable Set :— A set A is said to be countable if $\exists$ a map $\phi : N \to A$, which is one-one and onto.

$$\phi(N) \to Z$$

$$\phi(N) = \begin{cases} \frac{n}{2}, & \text{if } n = \text{even} \\ \frac{1-n}{2} & \text{if } n = \text{odd} \end{cases}$$

Ex- $B = \{1, 4, 9, \ldots\}$
$\phi : N \to B$
$\phi(n) = n^2$

Ex- $\phi = \{1, 2, 3, \ldots n\}$
$E = \{x_1, x_2, \ldots x_n\}$
$\phi(1) = x_1$
$\phi(2) = x_2$
$\phi(n) = x_n$

Member of elements in the set A is called Cardinality of set. It is denoted by |A| and defined by |A| = n

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Theorem:- Countable union of countable sets is countable.

Proof- Let $\{S_n : n \geq 1\}$ be a countable collection of countable sets and let $\boxed{S = \bigcup_{n=1}^{\infty} S_n}$. Then, we have show that S is countable.

Since S_1 is countable. Hence

$$S_1 = \{x_1^1, x_2^1, x_3^1, \ldots, x_n^1, \ldots\}$$
$$S_2 = \{x_1^2, x_2^2, x_3^2, \ldots, x_n^2, \ldots\}$$
$$\vdots$$
$$S_n = \{x_1^n, x_2^n, x_3^n, \ldots, x_n^n, \ldots\}$$

~~Since~~ Then, $S = \{\underbrace{x_1^1}_{1}, \underbrace{x_1^2, x_2^1}_{2}, \underbrace{x_1^3, x_2^2, x_3^1}_{3}, \ldots\}$

Since all the elements of S can be written as a in the form of one-one correspondens in the set of N.

Hence, S is countable.

18/08 **Theorem:-** Every infinite subset of countable set is countable.

Proof- Let A be a countable set, then
$$A = \{x_1, x_2, \ldots x_n, \ldots\}$$
where $x_i \neq x_j$

Let E be any infinite subset of A.
We claim that E is countable.

Let n be the smallest natural number s.t. $x_n \in E$.

Let $n_1 < n_2$ be the smallest natural number s.t. $x_{n_2} \in E$

Continuing for each k, Let n_k denote the smallest natural number bigger that n_{k-1} s.t. $x_{n_k} \in E$

$E = \{x_{n_1}, x_{n_2}, \dots x_{n_k}\}$

Define, $\phi : N \to E$

$\phi(k) = x_{n_k}$

Then ϕ is one-one & onto.

Hence E is countable.

Corollary- Countable union of finite set is countable.

Proof- Let $\{A_n\}_{n \in N}$ be the countable collection of finite sets then we have to show that $A = \bigcup_{n=1}^{\infty} A_n$ is countable.

$A_1 = \{\alpha_1^1, \alpha_2^1, \dots, \alpha_{n_1}^1\}$

$A_2 = \{\alpha_1^2, \alpha_2^2, \dots, \alpha_{n_2}^2\}$

$\vdots$

$A_n = \{\alpha_1^n, \alpha_2^n, \dots, \alpha_{n_m}^m\}$

Now $A = \bigcup_{n=1}^{\infty} A_n = \{\alpha_1^1, \alpha_1^2, \alpha_2^1, \ldots\}$

Since every arrow in the above diagram can be correspondence on the set of Natural Numbers.

Hence the set A is countable.
Otherwise It is finite set.
Since every finite set is countable therefore the finite set A is also countable.

Theorem :- Let A be a countable set and set $A^n = (A \times A \times A \times \cdots \times A)$ n times then A^n is countable for each $n \geq 1$ Prefer 1.

$n = 1$
$A^1 = A$
Since A is countable thus the result is hold here for $n = 1$.
Suppose A^{n-1} is countable
We claim that A^n is countable.
$A^n = A^{n-1} \times A$

$A^2 = A \times A$
$= \{(x, y) : x \in A, y \in A\}$
$A^3 = A^2 \times A$
$= \{(W, Z) : W \in A^2, Z \in A\}$

Let $x \in A^{n-1}$, then $\{(x, y) : y \in A\}$ is Countable
also $A^n = \bigcup_{x \in A^{n-1}} \{(x, y) : y \in A\}$ is countable.

Since countable union of countable set is countable Hence A^n is countable.

Question- Prove that the set of rational numbers is countable.

$\{\frac{p}{q} : p, q \in x$ provided $q = 0\} \subseteq A \times A$
$\subseteq A^2$

$\{(p,q) : p, q \in x$ provided $q \neq 0\}$

Since cartesian product of countable set is countable Hence the set of rational numbers $Q = \{(p,q), q \neq 0, p,q\}$ is countable.

OR

let $\{(p,q) : p, q \in$ set of $Q^+\}$

$$
\begin{array}{llll}
(1,1), & (1,2), & (1,3), & (1,4), \dots \\
(2,1), & (2,2), & (2,3), & \dots \\
(3,1), & (3,2), & (3,3), & \dots \\
(4,1), & (4,2), & (4,3), & \dots \\
\vdots & & & \\
(n,1), & (n,2), & (n,3), & \dots
\end{array}
$$

Since every element of Q^+ rational number can be written in one-one and onto corresponding to natural number. Hence Q^+ is countable.

Similarly we can say that set of all negative rational numbers can be countable.

Overall if the set of Q^+ and Q^- are countable, the set of all rational numbers is countable.

Question - Prove that the set of all seq^n of 0 and 1 is uncountable

Proof - Let $S = \{ (a_n)_{n=1}^{\infty} : a_n \in \{0,1\} \}$

then we claim that S is uncountable.

Let $S = \{ r_1, r_2, \ldots, r_n, \ldots \}$

Let if possible S be countable set.

Let,

$$r_1 = (a_{11}, a_{12}, a_{13}, \ldots a_{1n}, \ldots)$$
$$r_2 = (a_{21}, a_{22}, \ldots, a_{2n}, \ldots)$$
$$\vdots$$
$$r_m = (a_{m1}, a_{m2}, \ldots a_{mn}, \ldots)$$
$$\vdots$$

let $b = (b_1, b_2, \ldots b_n, \ldots)$

Where,

$$b_1 = \begin{cases} 1 & \text{if } a_{11} = 0 \\ 0 & \text{if } a_{21} = 1 \end{cases}$$

$\Rightarrow b \notin S$

which is contradiction.

Hence the set of all seq^n of 0 and 1 is uncountable.

Question- Perove that $A = (0, 1)$ is uncountable.

Let if Possible A is countable

$r_1 = (0, 0.a_{11}, 0.a_{12}, 0.a_{13} - - - , 0.a_n, - -)$

$r_2 = 0(a_{21}, a_{22}, a_{23}, - - a_{2n}, - -)$

$\vdots$

$r_n = 0(a_{n1}, a_{n2}, a_{n3}, - - - a_{nn}, - -)$

$\vdots$

Let $b = 0.b_1, b_2, \ldots, b_n, - - -$

$r_1 = a_{11} = 0.2873 - - -$ $(1, 2, - - 8)$

$r_2 = a_{22} = 0$

$b \neq r_n$ ~~for~~ any $n \geq 1$.

$\Rightarrow b \notin A$

Which is contradiction.

Hence the open interval $(0, 1)$ is uncountable.