Elementary Functions →

① Polynomial functions –

$$\omega = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = P(z)$$

where $a_0 \neq 0, a_1 \ldots a_n$ are complex constants and n is a positive integer called the degree of the polynomial $P(z)$.

② Rotational Algebraic Functions –

are defined as $\omega = \dfrac{P(z)}{Q(z)}$

$P(z), Q(z)$ are polynomials

$\omega = \dfrac{az+b}{cz+d}$ is called fractional linear transformation.

③ Exponential functions –

$$\omega = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

$$a^z = e^{z \ln a}$$

$a =$ real and positive

$\ln a$ is the natural logarithm of a.

Complex exponential functions have properties similar to those of real exponential function

for example $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$, $e^{z_1}/e^{z_2} = e^{z_1-z_2}$

④ Trigonometric functions –

$\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

$\sec z = \dfrac{1}{\cos z} = \dfrac{2}{e^{iz} + e^{-iz}}$, $\operatorname{cosec} z = \dfrac{1}{\sin z} = \dfrac{2i}{e^{iz} - e^{-iz}}$

$$\sin^2 z + \cos^2 z = 1$$

$$1 + \tan^2 z = \sec^2 z$$

$$1 + \cot^2 z = \operatorname{cosec}^2 z$$

$\sin(-z) = -\sin z$, $\cos(-z) = \cos z$

$\tan(-z) = -\tan z$.

Hyperbolic functions -

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}}$$

$$\operatorname{cosech} z = \frac{2}{e^z - e^{-z}}$$

$$\cosh^2 z - \sinh^2 z = 1$$

$$1 - \tanh^2 z = \operatorname{sech}^2 z$$

$$\coth^2 z - 1 = \operatorname{cosech}^2 z$$

$$\sin iz = i \sinh z, \quad \cos iz = \cosh z$$

$$\tan iz = i \tanh z, \quad \sinh iz = i \sin z$$

$$\cosh iz = \cos z, \quad \tanh iz = i \tan z$$

Logarithmic functions -

If $z = e^w$, then $w = \ln z$ — natural logarithmic of z.

thus the natural logarithmic function is the inverse of the exponential function and can defined as,

$$w = \ln z = \ln r + i(\theta + 2k\pi) \quad k = 0, \pm 1, \pm 2$$

Here z is $re^{i(\theta + 2k\pi)}$

$\ln z$ is the multiple-valued function the principle value or principle branch of $\ln z$ is sometimes defined as $\ln r + i\theta$ where $0 \le \theta \le 2\pi$ defined for the logarithmic function can be defined for real bases other than e. If $z = a^w$, $w = \log_a z$

Inverse Trigonometric functions -

If $z = \sin w$ then $w = \sin^{-1} z$ is called the inverse sign of z.

these functions, which are multiple-valued can be expressed in terms of natural logarithm as follows:

$\sin^{-1} z = \frac{1}{i}\left(\ln\left(iz + \sqrt{1-z^2}\right)\right)$

$\operatorname{cosec}^{-1} z = \frac{1}{i}\left(\ln\left[\frac{i + \sqrt{z^2-1}}{z}\right]\right)$

$\cos^{-1} z = \frac{1}{i}\left(\ln\left(z + \sqrt{z^2-1}\right)\right)$

$\sec^{-1} z = \frac{1}{i}\ln\left(\frac{1 + \sqrt{1-z^2}}{z}\right)$

Inverse Hyperbolic function →

If $z = \sinh w$, $w = \sinh^{-1} z$ is called the inverse hyperbolic sign of z.

$\sinh^{-1} z = \ln\left(z + \sqrt{z^2+1}\right)$

$\cosh^{-1} z = \ln\left(z + \sqrt{z^2-1}\right)$

$\operatorname{cosech}^{-1} z = \ln\left(\frac{1 + \sqrt{z^2-1}}{z}\right)$.

$\operatorname{sech}^{-1} z = \ln\left(\frac{1 + \sqrt{1-z^2}}{z}\right)$.