Equilibrium, Stability and Teeter Toy

Equilibrium of a particle :

An important special case of motion of a particle occurs when it is at rest or in equilibrium w.r.t an inertial frame of reference.

A necessary and sufficient condition for a particle to be at rest or in equilibrium is that the net force acting on the particle is zero which implies

$$\vec{F} = 0 \qquad \text{—— (1)}$$

If $\vec{F}$ is conservative then we can write

$$\vec{F} = -\vec{\nabla} U \qquad \text{—— (2)}$$

where U is a scalar potential.

$$\therefore \quad \vec{F} = -\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k} \right) = 0$$

$$\therefore \boxed{\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = \frac{\partial U}{\partial z} = 0} \qquad \text{—— (3)} \quad \text{Condition for equilibrium.}$$

Stability of Equilibrium

Stability of equilibrium is an important concept.
If a particle displaced slightly away from an equilibrium point P tends to return to its initial pt. P, then we call call P a point of stability or the stable pt. and the equilibrium is said to be a stable equilibrium.

A necessary and sufficient condition that an equilibrium to be stable is that the potential energy of the particle at the equilibrium point ~~must be minimum~~ must be minimum.

To check the stability of an equilibrium, we have to find out the second order derivative of the potential energy. If the 2nd order derivative is positive at that equilibrium point, then we say that the equilibrium is a stable equilibrium.

For example Consider the motion of a block attached to a spring on a ~~frictionalless~~ frictionless table. When the block is displaced from its equilibrium position and released it exibits S.H.M. When the spring is stretched or compressed, it will exert a force on the block which is given by $\vec{F} = -k\vec{x}$, where k is the spring constant and x is the displacement of the end of the spring from its equilibrium position. As a result the work is stored as elastic potential energy in the deformed spring and this is given by $U = \frac{1}{2}kx^2$, $k > 0$.

Let us now determine the equilibrium pt. and investigate the stability of equilibrium of this system.

For equilibrium the force is zero ie,

$$\vec{F} = \frac{dU}{dx} = 0$$

[since $\vec{F} = -\vec{\nabla}U \Rightarrow$ Conservative force field]

$$\text{or } \frac{d}{dx}\left(\frac{1}{2}kx^2\right) = 0$$

$$\therefore kx = 0 \Rightarrow x = 0 \quad \text{since } k > 0$$

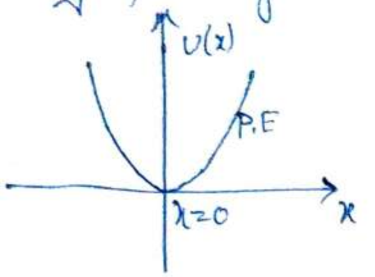
So equilibrium point is at $x = 0$.

To check the stability we have to find out the 2nd order derivative of the potential energy. If the 2nd order derivative turn out to be +ve then $x=0$ is a stable equilibrium pt.

$$\text{Here } \left.\frac{d^2U}{dx^2}\right|_{x=0} = k > 0 \quad \text{ie +ve}$$

$\therefore$ $x=0$ is a stable equilibrium point ie, when you displace the block from its equilibrium position $x=0$, it will tend to come back to its initial position $x=0$.

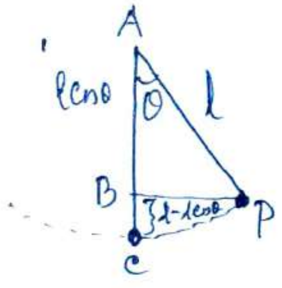
Graphically this can be explained.



From the graph it is obvious that at $x=0$, potential energy $U=0$ ie, minimum. So $x=0$ is a stable equilibrium because at stable equilibrium the potential energy must be minimum.

If an equilibrium has maximum potential energy (U) then that equilibrium is not stable.

Consider the example of a simple pendulum



Suppose a pendulum of length l and bob mass m is hanging at rest.
The potential energy at the bottom is taken to be zero.

Then at pt. P (as it swings through an angle θ with the vertical)

the potential energy

$$U(\theta) = mg(BC) = mg(AC - AB) = mg(l - l\cos\theta) = mgl(1-\cos\theta)$$

$$\boxed{\therefore U(\theta) = mgl(1-\cos\theta)}$$

At equilibrium $\frac{dU}{d\theta} = 0$

or, $\frac{d}{d\theta}\left(mgl(1-\cos\theta)\right) = 0$

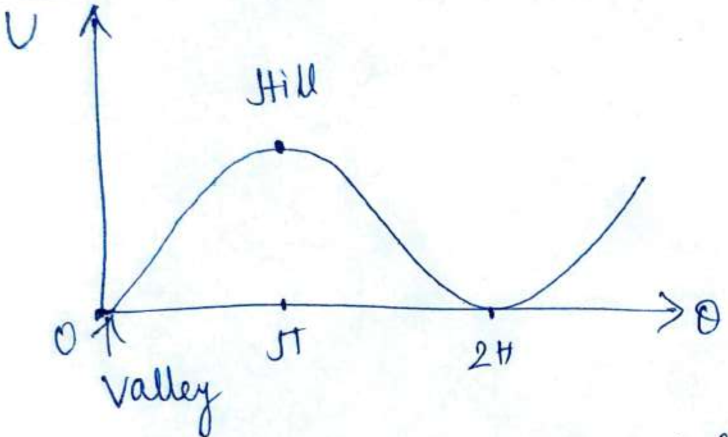
$\therefore mgl\sin\theta = 0 \Rightarrow \sin\theta = 0, \ \theta = 0 \text{ or } \pi.$

$\therefore$ The pendulum is in equilibrium for $\theta = 0$ and $\theta = \pi$.

At $\theta = 0$, the potential energy $U(\theta=0) = \underline{0}$
and at $\theta = \pi$, the potential energy $U(\theta = \pi) = \underline{2mgl}$

Thus we see at $\theta = \pi$, the potential energy has maximum value $U = 2mgl$ and the equilibrium is not stable.
So stable equilibrium occurs at $\theta = 0$.

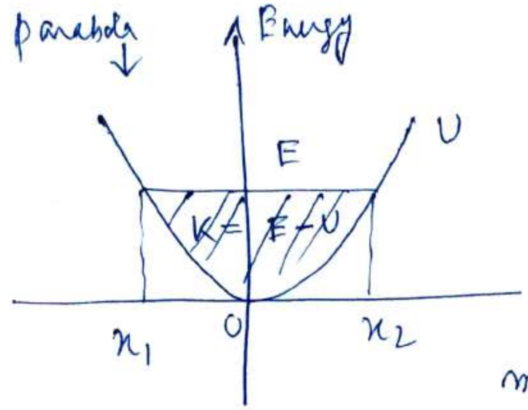
The potential energy curve



The minimum of the potential energy curve is a point of stable equilibrium and where U has maximum value that point is an unstable equilibrium.
The system is stable at the bottom of the potential energy (valley) and unstable at the top of the potential energy (hill).

Energy Diagrams

If we plot the total energy E and potential energy U as a function of position (x) then from the energy diagram we will get some interesting features of the motion of one dimensional system. The K.E. of the system $K = E - U$ can be found by inspection. Since the K.E. can not be -ve, the motion of the system is constrained to regions where $E \geq U$.



This is the energy diagram for a harmonic oscillator

[For harmonic oscillator
$F = -kx$ Eq^n of motion

$m\ddot{x} + kx = 0 \Rightarrow \ddot{x} + \frac{k}{m}x = 0$

or, $\ddot{x} + \omega^2 x = 0$ where $\omega = \sqrt{\frac{k}{m}}$ $\Big|\ \ddot{x} = \frac{d^2x}{dt^2}$

$K.E = \frac{1}{2}mv^2, \quad P.E = \frac{1}{2}kx^2$]

The potential energy of harmonic oscillator is $U = \frac{1}{2}kx^2$. The potential energy is a parabola centered at the origin. The harmonic oscillator force is a conservative force for which total energy is constant.

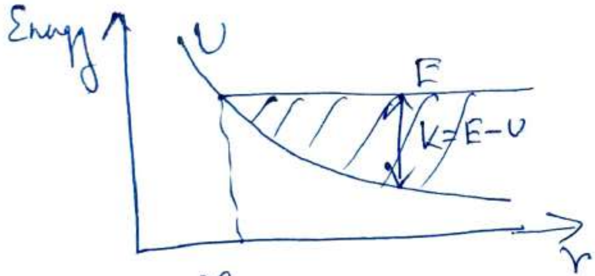
Since the total energy is constant, E is represented by a horizontal straight line. The motion is confined to the shaded region where $E \geq U$. The extreme two points (where K.E = 0 and total energy E = U) are known as turning points (x_1, x_2). The particle moves toward the origin with increasing K.E and the cycle is repeated.

The harmonic oscillator is a good example of bounded motion. As E increases the turning points move further and further off, but the particle can never move away freely. If E is decreased, the amplitude of motion decreases, until finally for $E=0$, the particle lies at rest at $x=0$

But we can find quite different behavior if U does not increase indefinitely with distance.

For instance, consider the case of a particle which is acted upon by a repulsive inverse square law force $\vec{F} = \frac{A}{r^2}\hat{r}$. Here potential energy $U = \frac{A}{r}$ where A is +ve.

The motion is constrained to a radial line.



There is a distance of closest approach r_{min}, as shown in the diagram but the motion is not bounded for large r since the potential energy U decreases with increase in distance.

If the particle moves toward the origin, it gradually loses its K.E. until it comes momentarily to rest at $r = r_{min}$ (where the total energy $E = P.E$ and $K.E = 0$). The motion then reverses and the particle moves out toward infinity.

Small oscillation in a bound system

Since all bound systems have a potential energy minimum at equilibrium, we can expect that all bound systems behave like a harmonic oscillator for small displacement (oscillation)

We can prove this with the help of Taylor's series expansion.

We know that a well behaved function $f(x)$ can be expanded in a Taylor's series about a pt. x. Thus

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \cdots$$

Suppose we expand the $U(r)$ of a bound system about the position r_0 where the potential is minimum. Then

$$U(r) = U(r_0) + U'(r_0)(r-r_0) + U''(r_0)\frac{(r-r_0)^2}{2!} + \cdots$$

$$= U(r_0) + (r-r_0)\left.\frac{dU}{dr}\right|_{r=r_0} + \frac{1}{2}(r-r_0)^2\left.\frac{d^2U}{dr^2}\right|_{r=r_0} + \cdots$$

However since U is a minimum at r_0, $\left.\frac{dU}{dr}\right|_{r_0} = 0$.

Furthermore, for sufficiently small displacement we can neglect the higher order terms. In that case

$$U(r) = U(r_0) + \frac{1}{2}(r-r_0)^2\left.\frac{d^2U}{dr^2}\right|_{r=r_0} \quad \text{--- (1)}$$

This the P.E. of a harmonic oscillator.

$$U(x) = \text{constant} + \frac{1}{2}kx^2$$

where we can identify the spring constant $k = \left.\frac{d^2U}{dr^2}\right|_{r=r_0}$

Energy and Stability – The Teeter Toy

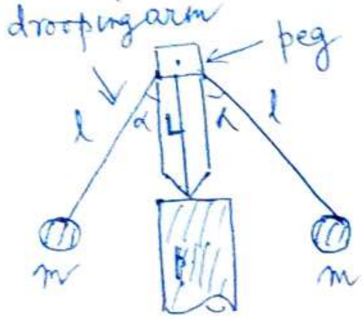


Fig ①

The teeter toy consists of two identical weights each having mass m which hang from a peg of length L on two drooping arms each having length l as shown in the figure. This arrangement is unpectedly stable – the toy can be spun or rocked with little danger of toppling over. We can see why this is so stable by looking at its potential energy.

Let us evaluate the potential energy of the teeter toy when it is given a tilt in the vertical plane through a small angle θ. As a result the teeter toy will perform S.H.M (Fig 2)

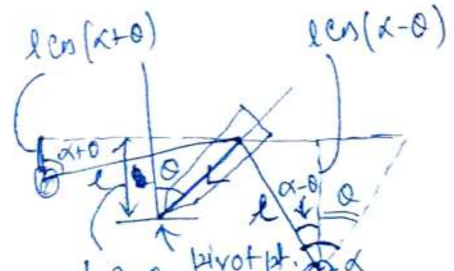


Fig (2)

We consider the gravitational potential energy zero at the pivot point. So the total potential energy of the masses of the teeter toy when it is given a small oscillation

$$U(\theta) = mg\left[L\cos\theta - l\cos(\alpha+\theta)\right] - mg\left[l\cos(\alpha-\theta) - L\cos\theta\right]$$

$$= mg\left[L\cos\theta - l\cos(\alpha+\theta)\right] + mg\left[L\cos\theta - l\cos(\alpha-\theta)\right]$$

$$= 2mgL\cos\theta - mgl\left[\cos(\alpha+\theta) + \cos(\alpha-\theta)\right]$$

using the identity $\cos(\alpha\pm\theta) = \cos\alpha\cos\theta \mp \sin\alpha\sin\theta$, we get

$$U(\theta) = 2mgL\cos\theta - mgl\left[2\cos\alpha\cos\theta\right]$$

$$\boxed{U(\theta) = 2mg\cos\theta\left[L - l\cos\alpha\right]}$$

[α is the angle between the drooping arm and the length of the peg]

Now equilibrium occurs when

$$\frac{dU(\theta)}{d\theta} = 0$$

or, $\frac{d}{d\theta}\left[2mg\cos\theta\left(L - l\cos\alpha\right)\right] = 0$

or, $-2mg\sin\theta\left(L - l\cos\alpha\right) = 0$

which implies $\sin\theta = 0$ ie $\theta = 0$ or $\pi/2$.

Now we will discard $\theta = \pi/2$ solution because we consider those values of θ which is less that $\pi/2$ (small oscillation)

So the solution is $\theta = 0$ as we expect from symmetry.

To investigate the stability of the equilibrium at equilibrium pt. $\theta = 0$, we must examine the second order derivative of the potential energy. We have

$$\frac{d^2U}{d\theta^2} = \frac{d}{d\theta}\left[-2mg\sin\theta\left(L - l\cos\alpha\right)\right]$$

$$= -2mg\cos\theta\left(L - l\cos\alpha\right) = 2mg\cos\theta\left(l\cos\alpha - L\right)$$

At equilibrium (at $\theta = 0$)

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta=0} = -2mg\cos\theta\left(L - l\cos\alpha\right)\Big|_{\theta=0} = -2mg\left(L - l\cos\alpha\right)$$

Now for stability of the equilibrium we require that 2nd order derivative must be +ve ie $\left.\frac{d^2U}{d\theta^2}\right|_{\theta=0} > 0$

$\therefore -2mg\left(L - l\cos\alpha\right) > 0$

or, $2mg\left(l\cos\alpha - L\right) > 0$

or, $\boxed{l\cos\alpha > L}$ since $2mg > 0$ always.

In order for the teeter toy to be stable, the weights must hang below the pivot point.

Example

In case of teeter toy, consider the mass m of each ball as 100 gm, $l = 50$ cm, $L = 10$ cm and $\alpha = 60°$. Find the frequency of small oscillation when the rod L is given a tilt in the plane of the paper by a small angle θ.

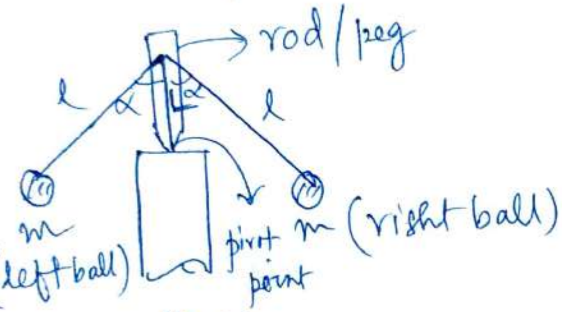


Fig ①

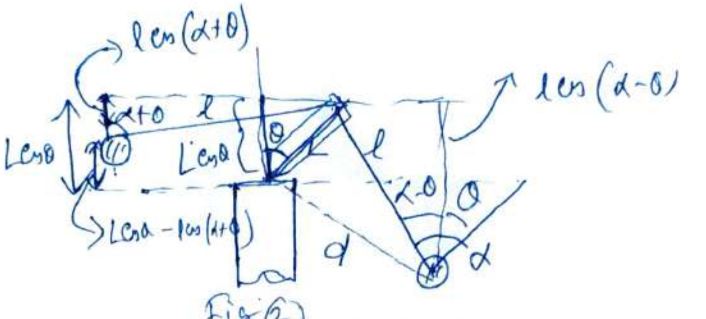


Fig ②

Let the potential energy at the pivot point be zero.
When the rod is given a tilt in the plane of the paper through a small angle θ then the left-ball ascends a height $[L\cos\theta - l\cos(\alpha+\theta)]$ and the right ball descends a height $[l\cos(\alpha-\theta) - L\cos\theta]$ [fig ②].

$\therefore$ The potential energy of the two masses

$$U(\theta) = 2mg\cos\theta\,(L - l\cos\alpha)$$

Equilibrium occurs $\dfrac{dU}{d\theta} = 0 \Rightarrow 2mg\sin\theta\,(L - l\cos\alpha) = 0$

$\therefore \sin\theta = 0 \Rightarrow \theta = 0$ is the equilibrium point.

For stable equilibrium at $\theta = 0$

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta=0} > 0$$

which gives $\boxed{l\cos\alpha > L} \Rightarrow$ stability condition

Now for small θ the potential energy of the teeter toy can be rewritten as

$$U(\theta) = 2mg\cos\theta\,(L - l\cos\alpha)$$

$$= 2mg(L - l\cos\alpha)\left(1 - \frac{\theta^2}{2!} + \cdots\right)$$

$$= -\underbrace{2mg(l\cos\alpha - L)}_{A \Rightarrow +\text{ve always}}\left(1 - \frac{\theta^2}{2}\right)$$ $\Rightarrow$ neglecting higher order terms as θ is very small

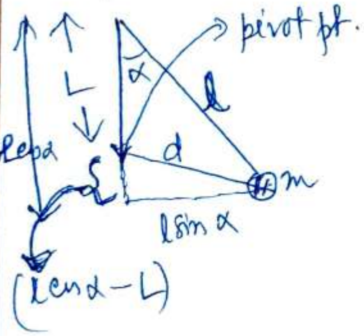
$$= -A\left(1 - \frac{\theta^2}{2}\right)$$ where $A = 2mg(l\cos\alpha - L) \Rightarrow$ +ve quantity.

$\therefore$ The total energy of the teeter toy is

$$E = P.E + K.E \text{ (of the two masses)}$$

$$= U(\theta) + \frac{1}{2}(2m)v^2 \qquad \left[v = d\dot{\theta} = d\underset{r}{\omega}\right]$$

Now $v = d\omega = d\dot{\theta}$ where d is the distance between the mass m and the pivot pt. P,



$$\therefore E = -A\left(1 - \frac{\theta^2}{2}\right) + \frac{1}{2}(2m)d^2\dot{\theta}^2$$

Since gravitational force is a conservative force field so the total energy (E) is constant.

$$\therefore \frac{dE}{dt} = 0$$

or, $$\frac{d}{dt}\left[\left(-A\left(1 - \frac{\theta^2}{2}\right) + \frac{1}{2}(2m)d^2\dot{\theta}^2\right)\right] = 0$$

or, $$\frac{A}{2}\,2\theta\dot{\theta} + \frac{1}{2}(2m)d^2\,2\dot{\theta}\ddot{\theta} = 0$$

$$\therefore A\theta\dot{\theta} + 2md^2\dot{\theta}\ddot{\theta} = 0$$

or, $$\ddot{\theta} + \frac{A}{2md^2}\theta = 0$$

or, $$\ddot{\theta} + \omega^2\theta = 0 \quad \text{where } \omega = \sqrt{\frac{A}{2md^2}}$$

∴ The frequency of oscillation is

$$\omega = \sqrt{\frac{A}{2md^2}} = \sqrt{\frac{g(l\cos\lambda - L)}{d^2}}$$

Here given $l = 0.5m$, $\cos\alpha = \cos 60^\circ = \frac{1}{2}$, $L = 0.1m$

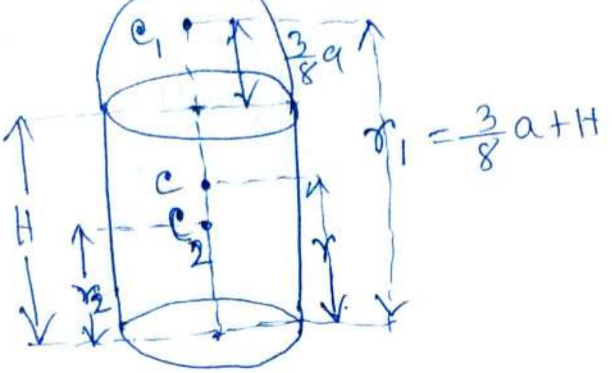
$g = 10m/sec^2$

and $d = \sqrt{(l\cos\alpha - L)^2 + (l\sin\alpha)^2} = 0.458\ m$

$\therefore \boxed{\omega = 2.673/sec}$ Ans.

Example

Find the centroid of a solid of constant density (σ) consisting of a cylinder of radius 'a' and height 'H' surmounted by a hemisphere of radius 'a'.



Let $\vec{r}$ be the distance of the c.o.M of the whole system from the base

We know that the C.M. of a solid hemisphere of radius 'a' $= \frac{3}{8}a$.

$\therefore$ The centroid (C_1) of the hemisphere of radius 'a' is at a distance $r_1 = \left(\frac{3}{8}a + H\right)$ from the base of the solid system.

$\therefore$ The mass of the hemisphere is $M_1 = V_1 \underset{\text{density}}{D} = \frac{2}{3}\pi a^3 \sigma$

We know that the C.M of the cylinder of height 'H' is at $H/2$ from the base of the solid system and its mass is

$$M_2 = V_2 D = \pi a^2 H \underset{\text{density}}{\sigma}$$

$\therefore$ The centroid distance ($\vec{r}$) is given by

$$\vec{r} = \frac{M_1 \vec{r}_1 + M_2 \vec{r}_2}{M_1 + M_2} = \frac{\left(\frac{2}{3}\pi a^3 \sigma\right)\left(\frac{3}{8}a + H\right) + \pi a^2 H \sigma \cdot \frac{H}{2}}{\frac{2}{3}\pi a^3 \sigma + \pi a^2 H \sigma}$$

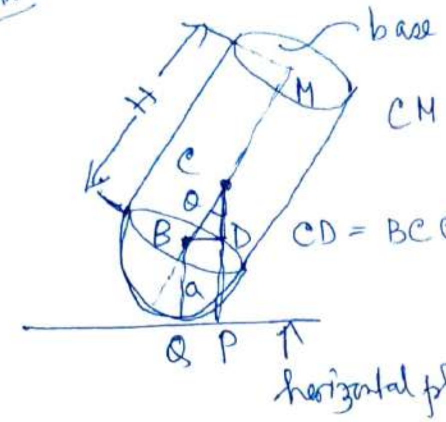
$$\boxed{r = \frac{3a^2 + 8aH + 6H^2}{8a + 12H}}$$

Ans.

Example

A uniform solid consists of a cylinder of radius 'a' and height 'H' and a hemisphere of radius 'a' as indicated in the figure. Prove that the solid is in stable equilibrium on a horizontal plane if and only if $a/H > \sqrt{2}$.

solution



$$CM = \frac{3a^2 + 8aH + 6H^2}{8a + 12H}$$ is the distance of the centroid C of the solid system from the base of the cylinder (see previous problem)

$CD = BC \cos\theta$

$CB = H - CM$

The centroid C of the system is at a distance CB from the centre B of the hemisphere given by

$$CB = H - CM = H - \frac{3a^2 + 8aH + 6H^2}{8a + 12H} = \frac{6H^2 - 3a^2}{8a + 12H}$$

Then the distance of the centroid C above the plane is

$$CP = CD + DP = CB\cos\theta + BQ$$
$$= \left(\frac{6H^2 - 3a^2}{8a + 12H}\right)\cos\theta + a$$

∴ The potential energy is

$$U = Mg\left[\left(\frac{6H^2 - 3a^2}{8a + 12H}\right)\cos\theta + a\right]$$

Equilibrium condition: $\frac{dU}{d\theta} = 0$

$$\text{or, } \frac{d}{d\theta}\left[Mg\left\{\left(\frac{6H^2 - 3a^2}{8a + 12H}\right)\cos\theta + a\right\}\right] = 0$$

$$\text{or, } -Mg\left(\frac{6H^2 - 3a^2}{8a + 12H}\right)\sin\theta = 0$$

$$\text{or, } Mg\left(\frac{3a^2 - 6H^2}{8a + 12H}\right)\sin\theta = 0 \Rightarrow \sin\theta = 0$$

ie $\theta = 0$

So the equilibrium is at $\theta = 0$.
Now we have to find out the stability of this equilibrium.
We know that equilibrium will be stable if

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta=0} > 0$$

$$\therefore \left.\frac{d^2U}{d\theta^2}\right|_{\theta=0} = \left. Mg\left(\frac{3a^2-6H^2}{8a+12H}\right)\cos\theta\right|_{\theta=0} > 0 \Rightarrow Mg\left(\frac{3a^2-6H^2}{8a+12H}\right) > 0$$

ie $3a^2-6H^2 > 0$ or, $\boxed{\frac{a}{H} > \sqrt{2}}$, proved.