## Solved Problems.

P.1) Expand $x \sin x$ as Fourier series in $(0, 2\pi)$

Soln. let $f(x) = x \sin x$

The Fourier series of $f(x)$ in $(0, 2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{---} (1)$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x\,dx$

$$= \frac{1}{\pi}\left[-x\cos x + \sin x\right]_0^{2\pi} = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\,dx \quad \text{---} (1)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cdot(2\cos nx \sin x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x \sin(n+1)x\,dx - \int_0^{2\pi} x \sin(n-1)x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left| -x\frac{\cos(n+1)x}{(n+1)} + \frac{\sin(n+1)x}{(n+1)^2}\right|_0^{2\pi} - \left| -\frac{x\cos(n-1)x}{(n-1)} + \frac{\sin(n-1)x}{(n+1)^2}\right|_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi\cos 2(n+1)\pi}{(n+1)} + 0 - 0 - 0 + \frac{2\pi\cos 2(n-1)\pi}{n-1}\right]$$

$$a_n = \frac{-1}{(n+1)} + \frac{1}{(n-1)} = \frac{2}{n^2-1}, \quad n \neq 1$$

For $n=1$ $\quad a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x\,dx$ by (1)

$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x\,dx = \frac{1}{2\pi}\left| -\frac{x\cos 2x}{2} + \frac{\sin 2x}{2}\right|_0^{\pi}$$

$$a_1 = \frac{1}{2\pi}\left[-\frac{2\pi}{2}\cos 2\pi + 0\right] = -\frac{1}{2}$$

and $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,(2\sin nx\sin x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\,dx\right]dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\cos(n-1)x\,dx - \int_0^{2\pi} x\cos(n+1)x\,dx\right]$$

$$= \frac{1}{2\pi}\left(\left|\frac{x\sin(n-1)x}{(n-1)} + \frac{\cos(n-1)x}{(n-1)^2}\right|_0^{2\pi} - \left|\frac{x\sin(n+1)x}{(n+1)} + \frac{\cos(n+1)x}{(n+1)^2}\right|_0^{2\pi}\right)$$

$$= \frac{1}{2\pi}\left[0 + \frac{1}{(n-1)^2} - \frac{1}{(n-1)^2} - 0 + \frac{1}{(n+1)^2} - \frac{1}{(n+1)^2}\right] = 0$$

Therefore the Fourier series expansion of the given function

$$x\sin x = -1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{1}{(n^2-1)}\cos nx$$

Ans.

P.2 Find the Fourier series of the function

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

and hence prove that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Soln. The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \quad (1)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\,dx + \int_0^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi^2 + \pi^2/2\right] = -\pi/2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx + \int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left|x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right) = \frac{1}{\pi n^2}\left((-1)^n - 1\right) = \begin{cases} \frac{-2}{n^2\pi}, & n \text{ is odd} \\ 0, & n \text{ is even}\end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{\pi\cos nx}{n}\right|_{-\pi}^{0} + \left|-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\left(1-(-1)^n\right) - \frac{\pi(-1)^n}{n}\right] = \frac{1}{n}\left[1 - 2(-1)^n\right]$$

Putting these values in ①, we get

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right) + \sum_{n=1}^{\infty}\left\{\frac{1-2(-1)^n}{n}\right\}\sin nx$$

Putting $x = 0$

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) \quad \text{—} ②$$

It is obvious that 0 is point of discontinuity for $f(x)$ so, we have.

$$f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right] = \frac{1}{2}\left[-\pi + 0\right] = -\pi/2$$

i.e. $f(0) = -\pi/2$

using in ② we get

$$-\pi/2 = -\pi/4 - 2/\pi\left(1/1^2 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\Rightarrow 1/1^2 + 1/3^2 + 1/5^2 + \cdots = \pi^2/8$$