Fourier Series.

Dirichlet's Conditions:

Any function $f(x)$ can be expanded as a Fourier Series $\frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{l}\right) + \sum b_n \sin\left(\frac{n\pi x}{l}\right)$

where a_0, a_n, b_n are constants, provided
(i) $f(x)$ is periodic, single valued and finite
(ii) $f(x)$ has finite number of discontinuities in any one period
(iii) $f(x)$ has at most a finite number of maxima and minima

The conditions (i), (ii) & (iii) are known as Dirichlet's conditions.

The complete study of Fourier Series as per syllabus can be summarized according to the given intervals as
(a) In the interval of form $(0, 2l)$
(b) In the interval of form $(-l, l)$
(c) In half range

(a) In $(0, 2l)$

Let the period of the given function $f(x)$ is $2l$ then its fourier series expansion in $(0, 2l)$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_0^{2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

and $b_n = \frac{1}{l}\int_0^{2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

(b) In $(-l, l)$

The fourier series of $f(x)$ with period $2l$ in $(-l, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

and $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Note - In this case if $f(x)$ is odd then $a_0 = a_n = 0$
and if $f(x)$ is even then $b_n = 0$

(c) Half range series in $(0, l)$

(i) Half range fourier cosine series in $(0, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_{0}^{l} f(x)\,dx$

and $a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

(ii) Half range fourier sine series in $(0, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Function having points of dis continuity

In general, to determine values of a_0, a_n & b_n it was assumed that $f(x)$ is continuous. even the function which has finite number of points of finite dis continuity, is expressible as a fourier series.

Suppose $f(x)$ has $x = c$ as point of dis continuity in $(0, 2l)$ ~~then~~ and $f(x)$ is defined by.

$$f(x) = \begin{cases} \phi(x) & 0 < x < c \\ \psi(x) & c < x < 2l \end{cases}$$

Then

$$a_0 = \frac{1}{l}\left[\int_0^c \phi(x)\,dx + \int_c^{2l} \psi(x)\,dx \right]$$

$$a_n = \frac{1}{l}\left[\int_0^c \phi(x) \cos\left(\frac{n\pi x}{l}\right) dx + \int_c^{2l} \psi(x) \cos\left(\frac{n\pi x}{l}\right) dx \right]$$

and $$b_n = \frac{1}{l}\left[\int_0^c \phi(x) \sin\left(\frac{n\pi x}{l}\right) dx + \int_c^{2l} \psi(x) \sin\left(\frac{n\pi x}{l}\right) dx \right]$$

and at the point of dis continuity i.e, at $x = c$ the value of $f(x)$ is the A.M. of left hand and right hand limits at $x = c$

i.e. $$f(c) = \frac{1}{2}\left[f(c+0) + f(c-0) \right]$$

Ex 2 Find the Fourier series of function

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi \\ 2\pi - x & \text{for } \pi \le x \le 2\pi \end{cases}$$

Soln: Here the given interval is $(0, 2\pi)$ so $l = \pi$ then the fourier series of the given function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{——— (1)}$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$

$$= \frac{1}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\pi} + \left|\frac{(2\pi - x)^2}{-2}\right|_{\pi}^{2\pi}\right] = \pi$$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$

$$= \frac{1}{\pi}\left[\left|x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right|_0^{\pi} + \left|(2\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{\cos n\pi}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} -4/\pi n^2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

and

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right|_0^{\pi} + \left|-(2\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right|_{\pi}^{2\pi}\right]$$

$$= 0$$