Fourier Transform

The Fourier transform of function $F(x)$ is denoted by $f\{F(x)\}$ or $f(P)$ and is defined by

$$f\{F(x)\} = \int_{-\infty}^{\infty} F(x)\, e^{ipx}\, dx \quad (1)$$

and the inverse fourier transform of $f(P)$ is given by

$$F^{-1}\{f(P)\} = F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(P)\, e^{-ipx}\, dp \quad (2)$$

It is called inversion formula corresponding (1)

Fourier Sine Transform of $F(x)$ in $0<x<\infty$

$$F_s\{F(x)\} = f_s(P) = \int_0^{\infty} F(x) \sin px\, dx \quad (3)$$

Its inversion formula is

$$F(x) = F_s^{-1}\{f_s(P)\} = \frac{2}{\pi}\int_0^{\infty} f_s(P) \sin px\, dx \quad (4)$$

Fourier cosine Transform of $F(x)$ in $0<x<\infty$

$$F_c\{F(x)\} = f_c(P) = \int_0^{\infty} F(x) \cos px\, dx \quad (5)$$

Its inversion formula is

$$F(x) = F_c^{-1}\{f_c(P)\} = \frac{2}{\pi}\int_0^{\infty} f_c(P) \cos px\, dx \quad (6)$$

Properties of Fourier Transform

(1) Linear Property

$$F\{c_1 F_1(x) \pm c_2 F_2(x) \pm \cdots\} = c_1 F\{F_1(x)\} \pm c_2 F\{F_2(x)\} \pm \cdots$$

(2) **Change of Scale property**

If $f(p)$ is the fourier transform of $F(x)$ then $F\{F(ax)\} = \frac{1}{a} f(p/a)$

(3) **Shifting property**

If $f(p)$ is the fourier transform of $F(x)$

then $F\{F(x-a)\} = e^{iap} f(p)$

and $F\{F(x+a)\} = e^{-iap} f(p)$

(4) **Modulation Theorem** - If $f(p)$ is the fourier transform of $F(x)$ then

$$F\{F(x)\cos ax\} = \frac{1}{2}[f(p+a) + f(p-a)]$$

Also if $f_s(p)$ and $f_c(p)$ fourier sine and cosine transforms of $F(x)$ then

(i) $F_s\{F(x)\cos ax\} = \frac{1}{2}[f_s(p+a) + f_s(p-a)]$

(ii) $F_c\{F(x)\sin ax\} = \frac{1}{2}[f_s(p+a) - f_s(p-a)]$

(iii) $F_s\{F(x)\sin ax\} = \frac{1}{2}[f_c(p-a) - f_c(p+a)]$

Problems

P.1 Find fourier transform of

$$f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Hence evaluate $\int_0^\infty \frac{\sin x}{x}\,dx$

Solution: The given function is

$$F(x) = \begin{cases} 1 & \text{for } |x| < 1 \quad \text{i.e. } -1 < x < 1 \\ 0 & \text{for } |x| > 1 \quad \text{i.e. } x > 1 \text{ or } x < \end{cases}$$

we know that

$$F\{F(x)\} = \int_{-\infty}^{\infty} F(x)\, e^{ipx}\, dx$$

$$= \int_{-\infty}^{-1} f(x) e^{ipx} dx + \int_{-1}^{1} f(x) e^{ipx} dx + \int_{1}^{\infty} f(x) e^{ipx} dx$$

(break interval of integration according given $f(x)$)

$$= \int_{-\infty}^{-1} 0 \cdot e^{ipx} dx + \int_{-1}^{1} 1 \, e^{ipx} dx + \int_{1}^{\infty} 0 \, e^{ipx} dx$$

$$= 0 + \left[\frac{e^{ipx}}{ip}\right]_{-1}^{1} + 0 = \frac{e^{ip} - e^{-ip}}{ip} = \frac{2\sin p}{p}$$

Now by inversion formula we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(p) e^{-ipx} dp$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin p}{p} e^{-ipx} dp = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

putting $x = 0$

$$\int_{-\infty}^{\infty} \frac{\sin p}{p} dp = \pi \quad \therefore \int_{0}^{\infty} \frac{\sin p}{p} dp = \pi/2$$

P-2 Find the fourier sine transform of $e^{-|x|}$

Hence show that $\int_0^{\infty} \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$, $m > 0$

Soln. - we have

$$f_s(p) = F_s\{f(x)\} = \int_0^{\infty} f(x) \sin px \, dx$$

$$= \int_0^{\infty} e^{-|x|} \sin px \, dx$$

$$= \int_0^{\infty} e^{-x} \sin px \, dx \quad \because x > 0$$

$$= \frac{p}{1+p^2} \quad \left(\because \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}\right)$$

And by its inversion formula, we have

$$\frac{2}{\pi} \int_0^{\infty} f_s(p) \sin px \, dp = f(x)$$

$$\Rightarrow \int_0^{\infty} \frac{p}{1+p^2} \sin px \, dp = \pi/2 \, e^{-x} \quad (x > 0)$$

$$\therefore \int_0^\infty \frac{x}{1+x^2} \sin mx \, dx = \frac{\pi}{2} e^{-m}, \quad m>0$$

NOTE — In some problems, we use the technique called differentiation under integral sign (Leibnitz Rule) to evaluate integrals involved in the corresponding Fourier transform. See following problems!

Problem-3 Find Fourier sine transform of $\frac{e^{-ax}}{x}$.

Solution. We know that

$$F_s\{F(x)\} = \int_0^\infty F(x) \sin px \, dx$$

$$\therefore F_s\left\{\frac{e^{-ax}}{x}\right\} = \int_0^\infty \frac{e^{-ax}}{x} \sin px \, dx = I, \text{say} \quad \text{—(1)}$$

Diff. (1) w.r.to p, we get

$$\frac{dI}{dp} = \frac{d}{dp}\int_0^\infty \frac{e^{-ax}}{x} \sin px \, dx = \int_0^\infty \frac{\partial}{\partial p} \frac{e^{-ax}}{x} \sin px \, dx$$

(by Leibnitz Rule

$$\frac{dI}{dp} = \int_0^\infty \frac{e^{-ax}}{x} \cdot x \cos px \, dx$$

$$= \int_0^\infty e^{-ax} \cos px \, dx =$$

$$\frac{dI}{dp} = \frac{a}{a^2+p^2} \quad \left(\because \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}\right)$$

Integrating, we get $I = \tan^{-1}(p/a) + c$ —(2)

from (1), when $p = 0 \Rightarrow I = 0$

applying it in (2) we get $c = 0$

therefore $I = F_s\left\{\frac{e^{-ax}}{x}\right\} = \tan^{-1}(p/a)$ Ans.

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