UNIT - IV

Dt. 09-10-23

Pg.

Function of Several Variable

Inner product ~ Let $x = (x_1, x_2, \ldots x_n) \in R^n$

$$y = (y_1, y_2, \ldots\ldots, y_n) \in R^n$$

then inner product of two vectors x and y denoted by $x \cdot y$ or $\langle x, y \rangle$ and defined by

$$x \cdot y = (x_1, x_2, x_3, \ldots x_n) \cdot (y_1, y_2, \ldots y_n)$$
$$= x_1 y_1 + x_2 y_2 + \ldots\ldots + x_n y_n$$
$$= \sum_{i=1}^{n} x_i y_i$$

The inner product is called dot product or scalar product.

Norm function ~ Let $x \in R^n$, then the norm of x is denoted by $\|x\|$ and defined by

$$\|x\| = \sqrt{\langle x, x \rangle}$$
$$= \sqrt{\sum_{i=1}^{n} x_i^2}$$

If $x \in R$ then $\|x\| = \sqrt{x \cdot x} = \sqrt{x^2} = |x| \geq$

$$\|\cdot\| : \mathbb{R}^n \longrightarrow \mathbb{R}$$

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* Proposition ~ The norm function i.e, $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$, satisfy the following condition

i) $\|x\| \geq 0$, $\forall x \in \mathbb{R}^n$

ii) $\|\alpha x\| = |\alpha| \cdot \|x\|$, $\forall \alpha \in \mathbb{R}$, $\forall x \in \mathbb{R}^n$

iii) $\|x+y\| \leq \|x\| + \|y\|$ $\forall$ $x, y \in \mathbb{R}^n$.

iv) $\big| \|x\| - \|y\| \big| \leq \|x-y\|$, $\forall$ $x, y \in \mathbb{R}^n$

Proof :

i) If $x \in R$, then $\|x\| = \sqrt{x \cdot x} = \sqrt{x^2}$
$= |x| \geq 0$

ii) $\alpha \in R$, $x \in \mathbb{R}^n$ $\Rightarrow$ $\alpha x \in \mathbb{R}^n$

$$\|\alpha x\| = \|(\alpha x_1, \alpha x_2, \alpha x_3, \cdots \alpha x_n)\|$$

$$= \sqrt{\alpha^2 x_1^2 + \alpha^2 x_2^2 + \cdots\cdots + \alpha^2 x_n^2}$$

$$= \sqrt{\alpha^2 (x_1^2 + x_2^2 + x_3^2 \cdots\cdots x_n^2)}$$

$$= |\alpha| \cdot \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$= |\alpha| \cdot \|x\|$$

iii)

Schwatiz lemma (inequality) : let $x, y \in \mathbb{R}^n$. then the following inequality hold

$$|x \cdot y| \leq \|x\| \cdot \|y\| \quad \forall\, x, y \in \mathbb{R}^n.$$

let $x = 0 \in \mathbb{R}^n$

$$x \cdot y = 0 \cdot y = 0$$

$$\|x\| = \|0\| = 0$$

for $x = 0$

$$|x \cdot y| \leq \|x\| \cdot \|y\| \quad \text{(holds)}$$

Next, if $x \neq 0$, $\|x\| > 0$
let

$$z = y - \frac{(x \cdot y)x}{\|x\|^2} \in \mathbb{R}^n$$

$$(z \cdot z) = \left(y - \frac{(x \cdot y)x}{\|x\|^2}\right)\left(y - \frac{(x \cdot y)x}{\|x\|^2}\right)$$

$$z \cdot z = y \cdot y + \frac{\{(x \cdot y)x\}}{\|x\|^2} \cdot \frac{\{(x \cdot y)x\}}{\|x\|^2} - 2\frac{(x \cdot y)(x \cdot y)}{\|x\|^2}$$

$$= \|y\|^2 + \frac{(x \cdot y)^2 (x \cdot x)}{\|x\|^4} - \frac{2(x \cdot y)^2}{\|x\|^2}$$

$$= \|y\|^2 + \frac{(x \cdot y)^2 \|x\|^2}{\|x\|^4} - \frac{2(x \cdot y)^2}{\|x\|^2}.$$

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$$= \|y\|^2 - \frac{(x\cdot y)^2}{\|x\|^2}$$

$\because \|z\|^2 \geq 0$

$$\|y\|^2 - \frac{|x y|^2}{\|x\|^2} \geq 0$$

$$\|x\|^2 \cdot \|y\|^2 \geq |x\cdot y|^2$$

$\Rightarrow |x\cdot y| \leq \|x\| \cdot \|y\| \quad \forall\, x, y \in \mathbb{R}^n.$

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iii) (Minkowskis inequality)

$\|x+y\| \leq \|x\| + \|y\|$, $|x\cdot y| \leq \|x\|\cdot\|y\|$

$$\begin{aligned}
\|x+y\|^2 &= (x+y)(x+y) \\
&= x(x+y) + y\cdot(x+y) \\
&= x\cdot x + x\cdot y + y\cdot x + y\cdot y \\
&= \|x\|^2 + 2x\cdot y + \|y\|^2 \\
&\leq \|x\|^2 + 2\cdot\|x\|\cdot\|y\| + \|y\|^2 \\
&\leq (\|x\| + \|y\|)^2 .
\end{aligned}$$

or

$$\|x+y\| \leq \|x\| + \|y\|$$

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iv) $\big| \|x\| - \|y\| \big| \leq \|x-y\|$, $\forall\ x, y \in \mathbb{R}$.

$\forall\ x, y \in \mathbb{R}^n$

$\because\ x = (x-y) + y$

$\|x\| = \cancel{\|x-y\|}\ \|(x-y) + y\|$

$\leq \|x-y\| + \|y\|$

$\|x\| - \|y\| \leq \|x-y\|$ —— (1)

Also, $y = (y-x) + x$

$\|y\| = \|(y-x) + x\|$

$\|y\| \leq \|y-x\| + \|x\|$

$\|y\| - \|x\| \leq \|y-x\|$ —— (2)

$-(\|x\| - \|y\|) = \|(-1)(x-y)\|$

$-(\|x\| - \|y\|) \leq \|x-y\|$ —— (2)

from eq (1) & (2), we get,

$\big| \|x\| - \|y\| \big| \leq \|x-y\|$

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Ques Prove that $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2+\|y\|^2)$

$\|x+y\|^2 + \|x-y\|^2$

$(x+y)\cdot(x+y) + (x-y)(x-y)$

$x\cdot(x+y) + y\cdot(x+y) + x(x-y) - y\cdot(x-y)$

$x\cdot x + \cancel{x\cdot y} + \cancel{y\cdot x} + y\cdot y + x\cdot x - \cancel{x\cdot y} - \cancel{y\cdot x} + y\cdot y$

$2\|x\|^2 + 2\|y\|^2$

$2(\|x\|^2 + \|y\|^2)$.

Ques Prove that $\|x+y\|^2 - \|x-y\|^2 = 4x\cdot y$

$= \|x+y\|^2 - \|x-y\|^2$

$= (x+y)(x+y) - [(x-y)(x-y)]$

$= \cancel{x\cdot x} + x\cdot y + y\cdot x + \cancel{y\cdot y} - [\cancel{x\cdot x} - x\cdot y - y\cdot x$
$+ \cancel{y\cdot y}$

$= 2x\cdot y + 2y\cdot x$

$= 4x\cdot y$

$\frac{a+b}{2} \geq \sqrt{ab}$

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<u>Remark</u> ~ $x \in \mathbb{R}^n$ $\quad x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots\dots x_n^2}$$

$$= \sqrt{|x_1|^2 + |x_2|^2 + \dots\dots |x_n|^2}$$

$$= \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

$$\rightarrow \quad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad 1 \leq p < \infty$$

$$\rightarrow \quad \|x\|_\infty = \sup_{n \geq 1} |x_n|$$

<u>Ques</u> Prove that $\|x+y\| \cdot \|x-y\| = \|x\|^2 + \|y\|^2$

: $\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$

: $\|x\|^2 + \|y\|^2 = \frac{1}{2}\left(\|x+y\|^2 + \|x-y\|^2\right)$

$$\geq \sqrt{\|x+y\|^2 \cdot \|x-y\|^2}$$

$\|x\|^2 + \|y\|^2 \geq \|x+y\| \cdot \|x-y\|$

Result ~ $\forall x \in (x_1, x_2, \ldots\ldots, x_n) \in \mathbb{R}^n$

$$\|x\| \leq \sum_{i=1}^{n} |x_i| \leq \sqrt{n} \cdot \|x\|$$

Proof ~

$e_1 = (1, 0, 0, \ldots\ldots)$

$e_2 = (0, 1, 0, \ldots\ldots)$

$\vdots$

$e_n = (0, 0, \ldots\ldots 1)$

$x = x_1 e_1 + x_2 e_2 + \ldots\ldots + x_n e_n$

$$x = \sum_{i=1}^{n} x_i e_i$$

$$\|x\| \leq \sum_{i=1}^{n} \|x_i e_i\| = \sum_{i=1}^{n} |x_i| \cdot \|e_i\|$$

$$= \sum_{i=1}^{n} |x_i| \quad \text{--- } (1)$$

Next,

let $y = (|x_1|, |x_2|, \ldots\ldots, |x_n|) \in \mathbb{R}^n$.

$z = (1, 1, 1, \ldots\ldots 1) \in \mathbb{R}^n$.

Then

$$\|y\| = \sqrt{|x_1|^2 + |x_2|^2 + \ldots\ldots |x_n|^2} = \|x\|$$

$$\|z\| = \sqrt{1 + 1 + \ldots\ldots + 1} = \sqrt{n}$$

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By Cauchy's Schwartz Inequality,

$$|y \cdot x| \leq \|y\| \cdot \|x\|$$

$$\sum_{i=1}^{n} |x_i| \leq \sqrt{n}\ \|x\| \quad \text{—— ②}$$

By ① and ②,

$$\|x\| \leq \sqrt{n} \cdot \|x\|$$

* Linear Transformation

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a function. The function T defined on $\mathbb{R}^n$ to $\mathbb{R}^n$ is said to be linear transformation if

$$T(ax + by) = aT(x) + bT(y)$$
$$\forall\ x, y \in \mathbb{R}^n, \quad \forall\ a, b \in \mathbb{R}.$$

* Proposition

→ Identity and zero transformation on $\mathbb{R}^n$ into $\mathbb{R}^n$ are L.T.

Proof~ $I : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$I(x) = x, \quad \forall\ x \in \mathbb{R}^n$

$$T(x+y) = x + y$$
$$= I(x) + I(y) \quad . \forall\ x \in \mathbb{R}^n$$

$\forall\ x \in \mathbb{R}^n$ and $a \in \mathbb{R}$

$$I(ax) = ax = a\,I(x)$$

Hence the identity transformation I is L.T.

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→ Next, $O : \mathbb{R}^n \longrightarrow \mathbb{R}^n$
defined by $O(x) = 0 \quad \forall \; x, y \in \mathbb{R}^n$

$O(x+y) = 0 = 0+0 = O(x) + O(y)$

$\forall \; x \in \mathbb{R}^n$ and $a \in \mathbb{R}$

$O(ax) = 0 = a \cdot 0 = a \cdot O(x)$

Hence zero transformation is L.T.

→ $T_c : \mathbb{R}^n \rightarrow \mathbb{R}^m$
defined by $T_c(x) = c \neq 0, \quad \forall \; x \in \mathbb{R}^n$.

$\forall \; x, y \in \mathbb{R}^n$
$T_c(x+y) = c \neq c + c$
$= T_c(x) + T_c(y)$

Hence the non-zero constant transformation is not L.T.

Ques: Let $T : \mathbb{R}^n = \mathbb{R}$ be a L.T, Then for $x \in \mathbb{R}^n$, $\exists \; y \in \mathbb{R}^n$ such that $T(x) = y \cdot x$

Proof: $\because \; T : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a L.T.

$T(x) = Ax$, where $A = [a_{ji}]_{1 \times n}$

$$T(x) = Ax = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

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$= x_1 y_1 + x_2 y_2 + \cdots\cdots + x_n y_n$

$= (x_1, x_2, \cdots\cdots, x_n) \cdot (y_1, y_2, \cdots, y_n)$

$= x \cdot y$

$= y \cdot x$

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* Proposition ~ If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a L.T, then $\|T(x)\| \le M \|x\|$, $\forall\ x \in \mathbb{R}^n$. where M is a real constant.

Proof~ Let $\{e_i : 1 \le i \le n\}$ be a basis set of $\mathbb{R}^n$

where

$$e_i = (0, 0, \cdots\cdots, 0, \underset{\text{ith place}}{1}, 0, \cdots\cdots 0)$$

and

$\{u_j : 1 \le j \le m\}$ be a basis set of $\mathbb{R}^m$

where

$$u_j = (0, 0, \cdots, 0, \underset{\text{jth place}}{1}, 0, \cdots\cdots 0)$$

$\forall\ x \in \mathbb{R}^n$

$$x = \sum_{i=1}^{n} x_i e_i \qquad [T: \mathbb{R}^n \to \mathbb{R}^m]$$

$$T(x) = \sum_{i=1}^{n} x_i\ T(e_i)$$

$$= \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{m} a_{ji} u_j\right)$$

$$= \sum_{j=1}^{m} u_j \left(\sum_{i=1}^{n} a_{ji} x_i\right)$$

Taking norm,

$$\|T(x)\| = \left\| \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ji} x_i\right) u_j \right\|$$

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$$\le \sum_{j=1}^{m} \left(\sum_{i=1}^{n} |a_{ji}| \cdot |x_i| \right) \|u_j\| \quad \left[\because \|u_j\| = 1 \right]$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} |a_{ji}| \, |x_i| \right)$$

$$\left[\because |x_i| \le \|x\| \right]$$

$$\le \sum_{j=1}^{m} \left(\sum_{i=1}^{n} |a_{ji}| \right) \|x\|$$

$$\le M \cdot \|x\|$$

where $M = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ji} \right)$

* Open subset of $\mathbb{R}^n$ ~ for $x \in \mathbb{R}^n$, $r > 0$

$B(x, r) = \{ y \in \mathbb{R}^n : \|y - x\| < r \}$
is called open ball of $\mathbb{R}^n$ with centre x and radius r.
$B(x, r)$ is also called r-nbd of x.

$B(x, r)$ is also denoted as $B_r(x)$ or $N_r(x)$.

The punctured nbd of x is denoted by $B^*(r, x)$ or $N_r^*(x)$ and defined by

$$B^*(r, x) = B(r, x) - \{x\}$$

→ A subset S of $\mathbb{R}^n$ is called an open subset of $\mathbb{R}^n$ if for any $x \in S$, for some $r > 0$, there exist an open ball $B(x, r)$ such that

$$B(x, r) \subset S$$

$T(x+y) = T(x) + T(y)$
$T(ax) = a\,T(x)$

* Addition & Multiplication of two L.T. Pg.

Definition ~ Let S be a subset of $\mathbb{R}^n$.
Let $f, g : S \to \mathbb{R}^m$ be two transformation defined

$$f+g : S \to \mathbb{R}^m \quad \text{and} \quad af : S \to \mathbb{R}^m$$
where $a \in \mathbb{R}$

by
$$(f+g)x = f(x) + g(x), \quad \forall\, x \in S$$
$$(af)x = af(x), \quad \forall\, x \in S \text{ and } a \in \mathbb{R}$$

* Theorem ~ Let $L, T : \mathbb{R}^n \to \mathbb{R}^m$ be two L.T., then show that
$L+T : \mathbb{R}^n \to \mathbb{R}^m$ and $(aL) : \mathbb{R}^n \to \mathbb{R}^m$
are also L.T.

$$\begin{aligned}(L+T)(x+y) &= L(x+y) + T(x+y)\\ &= L(x) + L(y) + T(x) + T(y)\\ &= L(x) + T(x) + L(y) + T(y)\\ &= (L+T)(x) + (L+T)(y)\end{aligned}$$

$$\begin{aligned}(L+T)\,ax &= \cancel{a\,(L+T)(x)} = L(ax) + T(ax)\\ &= aL(x) + aT(x)\\ &= a(L(x) + T(x))\\ &= a\,(L+T)(x)\end{aligned}$$

~~Hence L+T is L.T.~~

$$\begin{aligned}(aL)(x+y) &= a\,L(x+y)\\ &= a\,[L(x) + L(y)]\\ &= (aL)(x) + (aL)(y)\end{aligned}$$

$$\begin{aligned}(aL)(bx) &= a\,L(bx)\\ &= ab\,L(x)\\ &= b\,(aL)(x)\end{aligned}$$

Hence (aL) is L.T.

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* Limit ~ Let $f: R \to R$ be a function and $a \in \mathbb{R}$ then

$$|f(x) - l| < \varepsilon \quad , \text{whenever } |x-a| < \delta$$

Or

$$\lim_{x \to a} f(x) = l$$

Definition - Let S be a subset of R^n. Let $f: S \to R^m$ be a function if for every $\varepsilon > 0$, $\exists$ a number $\delta > 0$ such that $x \in B^*(a, \delta) \Rightarrow f(x) \in B(b, \varepsilon)$

then we say that the limit of $f(x)$ is $b \in \mathbb{R}^n$, when x ~~tends~~ to $a \in S$.

Equivalently, $f(x) \to b$, whenever $x \to a$.

Or

$$\lim_{x \to a} f(x) = b$$

Or

$$\|f(x) - b\| < \varepsilon, \text{ whenever } \|x-a\| < \delta.$$

Or

$$f(B^*(a, \delta)) \subseteq B(b, \varepsilon)$$