* Proposition ~ Let S be a subset of $\mathbb{R}^n$ and $f: S \to \mathbb{R}^m$ be a function then limit of $f(x)$ as x tends to a exist iff for every $\varepsilon > 0$, $\exists$ a number $\delta > 0$ such that

$$x, y \in B^*(a, \delta) \Rightarrow \|f(x) - f(y)\| < \varepsilon$$

Proof ~ Let $\lim\limits_{x \to a} f(x) = b$

For $\varepsilon > 0$, $\exists$ $\delta > 0$ such that

$$x \in B^*(a, \delta) \Rightarrow f(x) \in B\left(b, \frac{\varepsilon}{2}\right)$$

$$\Rightarrow \|f(x) - b\| < \frac{\varepsilon}{2}$$

Also, $y \in B^*(a, \delta) \Rightarrow \|f(y) - b\| < \frac{\varepsilon}{2}$

Now,

$$\|f(x) - f(y)\| = \|(f(x) - b) + (b - f(y))\|$$

$$\leq \|f(x) - b\| + \|b - f(y)\|$$

$$= \|f(x) - b\| + \|f(y) - b\|$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

Dt. ______

Pg. ______

* Theorem ~ let $f: S \to \mathbb{R}^m$ be two maps let $a \in S$ and $\lim_{x \to a} f(x) = b$,

$\lim_{x \to a} g(x) = c$, then

(i) $\lim_{x \to a} (f \pm g)x = b \pm c$

(ii) $\lim_{x \to a} (\lambda f)(x) = \lambda b$, $\lambda \in \mathbb{R}$.

Proof ~

i) $\because \lim_{x \to a} f(x) = b$

as

$\|f(x) - b\| < \varepsilon$, whenever $\|x-a\| < \delta$.

$\therefore x \in B^*(a, \delta_1) \Rightarrow f(x) \in B\left(b, \frac{\varepsilon}{2}\right)$ — ①

Now,

$\lim_{x \to a} g(x) = c$

$x \in B^*(a, \delta_2) \Rightarrow g(x) \in B\left(c, \frac{\varepsilon}{2}\right)$ — ②

choose $\delta = \min\{\delta_1, \delta_2\}$

$x \in B^*(a, \delta) \Rightarrow \|f(x) - b\| < \frac{\varepsilon}{2}$, $\|g(x) - c\| < \frac{\varepsilon}{2}$

Now,

$\|(f+g)(x) - (b+c)\| = \|(f(x) - b) + (g(x) - c)\|$

$\leq \|f(x) - b\| + \|g(x) - c\|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, whenever $\|x-a\| < \delta$

Dt. ______

Pg. ______

Hence $\lim_{x \to a} (f+g)(x) = b+c$

(ii) If $\lambda = 0$, $(\lambda f)(x) = 0$, $\lambda b = 0$

then result hold.

let $\lambda \neq 0$

$\because \lim_{x \to a} f(x) = b$

$x \in B^*(a, \delta) \Rightarrow f(x) \in B\left(b, \frac{\varepsilon}{|\lambda|}\right)$

i.e,

$\|f(x) - b\| < \frac{\varepsilon}{|\lambda|}$, when $\|x-a\| < \delta$

Now,

$$\|(\lambda f)(x) - \lambda b\| = \|\lambda f(x) - \lambda b\|$$

$$= \|\lambda (f(x) - b\|$$

$$= |\lambda| \cdot \|f(x) - b\|$$

$$< |\lambda| \cdot \frac{\varepsilon}{|\lambda|}$$

$$= \varepsilon$$

Hence, $\lim_{x \to a} (\lambda f)(x) = \lambda b$

Dt. 20-10-23

Pg.

Theorem ~ Let S be a subset of $\mathbb{R}^n$ and $f, g : S \to \mathbb{R}$ be two maps. Let

$$\lim_{x \to a} f(x) = b \quad , \quad \lim_{x \to a} g(x) = c$$

and

$$(fg)(x) = f(x)\,g(x) \quad \text{and} \quad \left(\frac{1}{g}\right)x = \frac{1}{g(x)}$$

provided $g(x) \neq 0, \quad \forall x \in S$

Then

$$\lim_{x \to a} (fg)(x) = bc \quad \text{and} \quad \lim_{x \to a} \left(\frac{1}{g}\right)(x) = \frac{1}{c}$$

provided, $c \neq 0$.

Since $\lim_{x \to a} f(x) = b$ and $\lim_{x \to b} g(x) = c$

$\therefore$ For $0 < \varepsilon < 1$, $\exists$ a $\delta > 0$ such that

$$|f(x) - b| < \frac{\varepsilon}{1 + |b| + |c|}, \text{ whenever } \|x - a\| < \delta \quad \text{— (1)}$$

$$|g(x) - c| < \frac{\varepsilon}{1 + |b| + |c|}, \text{ whenever } \|x - a\| < \varepsilon \quad \text{— (2)}$$

Also, $g(x) = (g(x) - c) + c$

or

$$|g(x)| = |(g(x) - c) + c|$$

$$< 1 + |c|, \text{ whenever } \|x - a\| < \delta \quad \text{— (3)}$$

Dt. ______

Pg. ______

Now,

$$|(fg)(x) - bc| = |f(x) \cdot g(x) - bc|$$

$$= |g(x)\ (f(x) - b) + b\,(g(x) - c)|$$

$$\leq |g(x)|\,|f(x) - b| + |b| \cdot |g(x) - c|$$

$$< (1 + |c|)\left(\frac{\varepsilon}{1 + |b| + |c|}\right) + \frac{|b| \cdot \varepsilon}{1 + |b| + |c|}$$

$$= \cancel{(1 + |b| + |c|)} \cdot \frac{\varepsilon}{\cancel{1 + |b| + |c|}}$$

$$= \varepsilon$$

Hence $\lim_{x \to a} (fg)(x) = bc$

(ii) $\lim_{x \to a} g(x) = c$

For $0 < \varepsilon < \frac{1}{|c|}$, such that

$|g(x) - c| < \frac{\varepsilon |c|^2}{2}$, whenever $\|x - a\| < \delta$ — (1)

or

$|g(x) - c| < \frac{|c|}{2}$, $0 < \|x - a\| < \delta$ — (2)

Also, $g(x) = (g(x) - c) + c$

Dt. ____

Pg. ____

$$|g(x)| = |(g(x) - c) + c|$$

$$\geq |c| - |g(x) - c|$$

$$> |c| - \frac{|c|}{2} = \frac{|c|}{2} \quad \text{--- (3)}, \text{ from (2)}$$

Now, $\left| \frac{1}{g(x)} - \frac{1}{c} \right| = \left| \frac{c - g(x)}{c \cdot g(x)} \right|$

$$= \frac{|g(x) - c|}{|c| \cdot |g(x)|}$$

$$< \frac{\varepsilon\, |c|^2}{2|c|} \times \frac{2}{|c|} \quad \left[\text{using (1) \& (3)}\right]$$

$$= \varepsilon$$

Hence,

$$\lim_{x \to a} \left(\frac{1}{g}\right)(x) = \frac{1}{c}$$

* Continous Map (or function)

Let S be a subset of $\mathbb{R}^n$ and $f: S \to \mathbb{R}^m$ be a map. The map f is said to be continous at a point $a \in S$ if for every $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$f(B_\delta(a, \delta)) \subset B(f(a), \varepsilon) \quad \text{--- (1)}$$

Dt.

Pg.

where $B_S(a,\delta) = B(a,\delta) \cap S$

i.e,

$$x \in B_S(a,\delta) \Rightarrow f(x) \in B(f(a),\varepsilon)$$

i.e,

$$\|x-a\| < \delta \;,\; x \in B_S(a,\delta) \Rightarrow \|f(x)-f(a)\| < \varepsilon$$

i.e,

$$\lim_{x\to a} f(x) = f(a)$$

If S be a open subset of $\mathbb{R}^n$ and a is an interior point of S then condition ① can be written as

$$f(B_S(a,\delta)) \subset B(f(a),\varepsilon)\,.$$

26-10-23

Proposition ~ If S be an open subset of R^n and $f: S \to R^m$ be a function. If for $x, y \in S$, $\varepsilon > 0$

$$\|f(y) - f(x)\| \le M\|y-x\|, \quad M > 0$$

Then f is continous.

Proof ~ For $\varepsilon > 0$, set $\delta = \frac{\varepsilon}{M} > 0$

$$\|f(y) - f(x)\| \le M\|y-x\|$$

$$< M\left(\frac{\varepsilon}{M}\right), \text{ whenever } \|y-x\| < \delta$$

$$= \varepsilon$$

Dt. ______
Pg. ______

$= \|y-x\| < \delta \Rightarrow \|f(y)-f(x)\| < \varepsilon$

$\Rightarrow$ f is continous at $x \in S$.

Since x is an arbitrary point S and f is continous at $x \in S$, therefore f is continous function.

27-10-23

Theorem ~ Every L.T $T: \mathbb{R}^n \to \mathbb{R}^m$ is continous.

Given that $T: \mathbb{R}^n \to \mathbb{R}^m$ is L.T.

Let $\{e_i : 1 \le i \le n\}$ be a basis set of $\mathbb{R}^n$

$\{u_j : 1 \le j \le m\}$ be a basis of $\mathbb{R}^m$.

Let $x \in \mathbb{R}^n \Rightarrow x = \sum_{i=1}^{n} x_i e_i$

Then $T(x) = T\left(\sum_{i=1}^{n} x_i e_i\right)$

$T(x) = \sum_{i=1}^{n} x_i T(e_i)$

$T(x) = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{m} a_{ji} \cdot u_j\right)$

$T(x) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} x_i a_{ji}\right) u_j$

Taking norm both side

$\|T(x)\| \le \sum_{j=1}^{m} \sum_{i=1}^{n} |x_i| |a_{ji}| \|u_j\|$ $\boxed{\because |x_i| \le \|x\|}$

Dt. ________
Pg. ________

$$\|T(x)\| \leq \sum_{j=1}^{m} \sum_{i=1}^{n} |a_{ji}| \cdot \|x\|$$

$$\|T(x)\| \leq M \cdot \|x\| \quad , \text{ where } M = \sum_{j=1}^{m} \sum_{i=1}^{n}$$

Then $\forall\ x, y \in \mathbb{R}^n$, $y-x \in \mathbb{R}^n$

Therefore

$$\|T(y-x)\| \leq M \|y-x\| \quad - \text{ from (2)} \quad M>0$$

or

$$\|T(y) - T(x)\| \leq M \|y-x\|$$

for $\varepsilon > 0$, set $\delta = \frac{\varepsilon}{M} > 0$

whenever $\|y-x\| < \frac{\varepsilon}{M} - \delta$

$$\|T(y) - T(x)\| < M \cdot \frac{\varepsilon}{M} - \varepsilon$$

whenever $\|y-x\| < \delta$

$\therefore$ T is continous at $x \in \mathbb{R}^m$.

Since x is arbitrary point of $\mathbb{R}^n$, therefore T is continous on $\mathbb{R}^n$.

Dt. ___________

Pg. ___________

* Theorem : Define $\eta : \mathbb{R}^n \to \mathbb{R}$ by

$\eta(x) = \|x\|$, $\forall$ $x \in \mathbb{R}^n$

Show that $\eta : \mathbb{R}^n \to R$ is continous.

$\forall x, y \in \mathbb{R}^n \Rightarrow y - x \in \mathbb{R}^n$

then, $|\eta(y) - \eta(x)| = | \|y\| - \|x\| | \le \|y - x\|$

For $\varepsilon > 0$, set $\delta = \varepsilon$

$|\eta(y) - \eta(x)| \le \|y - x\| < \varepsilon$, whenever $\|y - x\| < \delta = \varepsilon$

$\therefore$ $\eta : \mathbb{R}^n \to R$ is continous at $x \in \mathbb{R}^n$.

Therefore the norm function is continous on $\mathbb{R}^n$

* Theorem ~ Let $S : \mathbb{R}^n \times \mathbb{R}^n \to R$ be a function defined by $S(x, y) = x \cdot y = \langle x, y \rangle$, $\forall$ $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Then S is continous map.

For $\varepsilon > 0$,

Let $(u, v) \in B((x, y), \delta)$

Then, $\|u - x\| \le \|(u, v) - (x, y)\| < \delta$

$\|v - y\| \le \|(u, v) - (x, y)\| < \delta$

Dt. ________

Pg. ________

Also,

$$\|v\| = \|(v-y)+y\| \leq \|v-y\| + \|y\|$$

$$\leq \delta + \|y\|$$

$$\|S(u,v) - S(x,y)\| = \|uv - xy\|$$

$$= \|(u-x)v + x(v-y)\|$$

$$\leq \|u-x\| \cdot \|v\| + \|x\| \cdot \|v-y\|$$

$$< \delta(\delta + \|y\|) + \|x\| \cdot \delta$$

$$\|S(u,v) - S(x-y)\| < \delta(1 + \|x\| + \|y\|)$$

For $\varepsilon > 0$, Set δ such that $\delta < \dfrac{\varepsilon}{1+\|x\|+\|y\|}$

Now,

$$\|S(u,v) - S(x,y)\| < \cancel{(1 + \|x\| + \|y\|)} \cdot \frac{\varepsilon}{\cancel{1+\|x\|+\|y\|}}$$

$$\leq \varepsilon$$

Hence the map S is continuous on $\mathbb{R}^n \times \mathbb{R}^n$.

Theorem ~ Let S be an open subset of $\mathbb{R}^n$. If $f: S \to \mathbb{R}^m$ and $g: f(S) \to \mathbb{R}^p$ are continous at a and f(a) respectively, then gof is continous at $a \in S$.

Since g(x) is continous and f(a) therefore

For $\varepsilon > 0$, $\exists$ $r > 0$, s.t

$$\|g(f(x))' - g(f(a))\| < \varepsilon, \text{ whenever } \|f(x) - f(a)\| < r \quad \text{——— (1)}$$

Also, f is continous, therefore for $r > 0$: $\exists$ $\delta > 0$ such that

$$\|f(x) - f(a)\| < r, \text{ whenever } \|x - a\| < \delta \quad \text{— (2)}$$

From (1) and (2), we get

$$\|(gof)(x) - (gof)(a)\| < \varepsilon, \text{ whenever } \|x - a\| < \delta$$

Hence the map gof is continous at $a \in S$.

→ **Derivative in an open subset of $\mathbb{R}^n$ –**

Let $f: \mathbb{R} \to \mathbb{R}$ be a function if f is differentiable at any point x of an interval in $\mathbb{R}$ then for $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < \varepsilon, \text{ whenever } |h| < \delta$$

i.e, $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

$$\lim_{h \to 0} \left[\frac{|f(x+h) - f(x) - hf'(x)|}{h} \right] = 0$$

Let $T(h) = (f'(x))h$, then $T: \mathbb{R} \to \mathbb{R}$ is L.T.

$$\lim_{h \to 0} \left[\frac{|f(x+h) - f(x) - T(h)|}{h} \right] = 0$$

$f: \mathbb{R} \to \mathbb{R}$ is said to be differentiable at x in any interval of $\mathbb{R}$ if $\exists$ a L.T. $T: \mathbb{R} \to \mathbb{R}$, such that

$$\lim_{h \to 0} \left[\frac{|f(x+h) - f(x) - T(h)|}{|h|} \right] = 0$$

Definition ~ Let S be an open subset of $\mathbb{R}^n$ and $f: S \to \mathbb{R}^m$ is said to be differentiable or derivable at $x \in S$ if $\exists$ a L.T, $T: \mathbb{R}^n \to \mathbb{R}^m$, such that

$$\lim_{h \to 0} \left[\frac{\| f(x+h) - f(x) - T(h) \|}{\|h\|} \right] = 0$$

we write,

$$T = \frac{d}{dx} f(x) = f'(x)$$

This derivability is known as Frechit's derivative $f : S \to \mathbb{R}^m$.

31-10-23

* Chain Rule for Differentiation of a function of Several variable ~

Theorem ~ Let S be an open subset of $\mathbb{R}^n$ containing a point x_0. Let H be an open subset of $\mathbb{R}^m$ containing $f(s)$.

Let f be a function on S into $\mathbb{R}^m$ and g be a function on H into $\mathbb{R}^p$

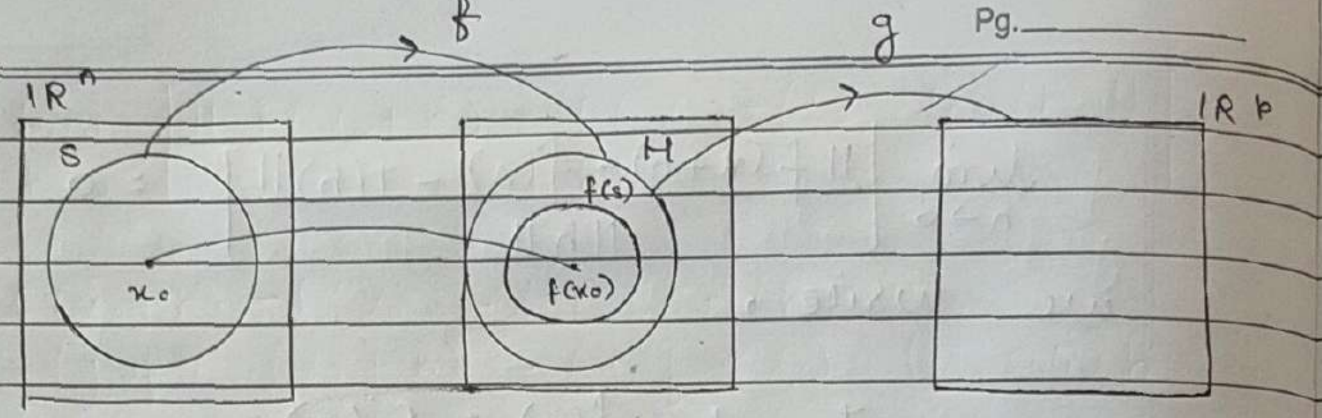
Let $F(x) = (g \circ f)(x) = g(f(x)), \forall x \in S$

If f is differentiable at x_0 and g differentiable at $f(x_0)$, then F is differentiable at x_0 and

$$F'(x_0) = g'(f(x_0)) \circ f'(x_0)$$

Dt. ____

Pg. ____



$S \subseteq \mathbb{R}^n$, $f(S) \subseteq H \subseteq \mathbb{R}^m$

Proof :- Let $f(x_0) = y_0$

Since f is diff at x_0, therefore $f'(x_0) = T_1$, where T_1 is the L.T. on $\mathbb{R}^n$ int $\mathbb{R}^m$.

Also, g is diff at $f(x_0)$, then $g'(f(x_0)) = T_2$, where T_2 is the L.T. on $\mathbb{R}^m$ into $\mathbb{R}^p$

i.e,

$$g'(y_0) = T_2$$

then,

$$\lim_{h\to 0}\left[\frac{\|f(x_0+h) - f(x_0) - T_1(h)\|}{\|h\|}\right] = 0$$

and

$$\lim_{k\to 0}\left[\frac{\|g(y_0+k) - g(y_0) - T_2(k)\|}{\|k\|}\right] = 0$$

Let $u(h) = f(x_0+h) - f(x_0) - T_1(h)$ —— ①

and $v(k) = g(y_0+k) - g(y_0) - T_2(k)$ —— ②

Dt.
Pg.

Define

$\eta_1(h)$ by $\eta_1(h) = \begin{cases} \frac{\|u(h)\|}{\|h\|}, & h \neq 0 \end{cases}$

and

$\eta_2(k)$ by $\eta_2(k) = \begin{cases} \frac{\|v(k)\|}{\|k\|}, & k \neq 0 \\ , & h = 0 \end{cases}$

Then,

$\|u(h)\| = \eta_1(h) \cdot \|h\|$, where $\eta_1(h) \to 0$ at $h \to 0$

and

$\|v(k)\| = \eta_2(k) \cdot \|k\|$, where $\eta_2(k) \to 0$ as $k \to 0$

Let $k = f(x_0 + h) - f(x_0)$ — (3)

$= u(h) + T_1(h)$, by using (1),

Then,

$$\|k\| = \|u(h) + T_1(h)\|$$
$$\leq \|u(h)\| + \|T_1(h)\|$$
$$= \eta_1(h) \cdot \|h\| + \|T_1\| \cdot \|h\|$$

$(\because \|T_1(h)\| \leq \|T_1\| \cdot \|h\|$, T_1 is L.T.)

or

$\|k\| \leq (\eta_1(h) + \|T_1\|) \|h\|$ — (4)

we have to show that,

$F'(x_0) = g'(f(x_0)) \circ f'(x_0)$
$= T_2 \circ T_1$

Dt.

Pg.

$$\lim_{h\to 0}\left[\frac{\|F(x_0+h)-f(x_0)-(T_2\circ T_1)(h)\|}{\|h\|}\right]=0$$

$f(x_0+h)-f(x_0)-(T_2\circ T_1)(h)$

$= (g\circ f)(x_0+h)-(g\circ f)(x_0)-(T_2\circ T_1)h$

$= g(f(x_0+h))-g(f(x_0)-T_2(T_1(h))$

$= g(f(x_0+h))-g(y_0)-T_2(T_1(h))$

$= g(f(x_0+h))-g(y_0)-T_2(T_1(h))$

$= v(k)+T_2(k)-T_2(T_1(h))$, By ② & ③.

$= v(k)+T_2(k-T_1h)$

$= v(k)+T_2(u(h))$, by ① and ③.

Now,

$\|f(x_0+h)-f(x_0)-(T_2\circ T_1)(h)\| \leq$
$\|v(k)\|+\|T_2(u(h))\|$

$\leq \|v(k)\|+\|T_2\|\cdot\|u(h)\|$

$\leq \eta_2(k)\cdot\|k\|+\|T_2\|\cdot\eta_1(h)\cdot\|h\|$

$\leq \eta_2(k)\left[\eta_1(h)+\|T_1\|\right]\|h\|+$
$\|T_2\|\,\eta_1(h)\cdot\|h\|$, by ④.

or

$$\lim_{h\to 0}\left[\frac{\|F(x_0+h)-f(x_0)-(T_2\circ T_1)(h)\|}{\|h\|}\right]\leq$$

Dt.

Pg.

$$\lim_{h\to 0} \eta_2(k)\left[\eta_1(h) + \|T_1\|\right] + \|T_2\|\cdot\|\eta_1(h)\|$$

Since, $\eta_1(h) \to 0$, as $h \to 0$

$\eta_2(k) \to 0$ as $k \to 0$

but $k \to 0$ as $h \to 0$ by (4),

Then

$\eta_2(k) \to 0$ as $h \to 0$

Thus

$$\lim_{h\to 0}\left[\frac{\|f(x_0+h) - f_0(x_0) - (T_2 \circ T_1)(h)\|}{\|h\|}\right] = 0$$

Hence

$$F'(x_0) = T_2 \circ T_1$$

$$= g'(f(x_0)) \circ f'(x_0)$$