Theorem 8 - "Every non-empty bounded below subset of $\mathbb{R}$ has an infimum in $\mathbb{R}$.

Proof → let S be a non-empty bounded below subset of $\mathbb{R}$,

Then, $\exists$ a real number 'k' s.t.

$x \geq k \;\; \forall \; x \in S$ ---- (1)

let us define a subset T of $\mathbb{R}$ by

$T = \{y : y = -x, \; x \in S\}$

Since S is non-empty, so is the set T.

Further from eq ①, we can say

$-x \leq -k$

$y \leq -k \quad \forall \; y \in T$

Thus, T is bounded above by $-k$.
Hence, by Completeness Axiom, T has a supremum in R.

Let $\alpha = \sup T$

Since, α is an upper bound of T
$\Rightarrow -\alpha$ is a lower bound of S.

We need to show that $-\alpha$ is the Inf S.
Let l be any lower bound of S.
Then $-l$ is an upper bound of T.
Since $\sup T = \alpha$

$\Rightarrow \alpha \leq -l$

$\Rightarrow -\alpha \geq l$

Hence $-\alpha$ is the infimum of S.

Proved