(v) A set which is both open & closed
31/8/22 $\phi \rightarrow$ Null set

Theorem→ A subset E of R is closed iff its complement E^c is open.

Proof→ Suppose that E is closed. let x be an arbitrary element of E^c so that $x \notin E$.
Since, E is closed & $x \notin E$
$\Rightarrow x$ cannot be a limit point of E.
So, $\exists$ a nbd 'N' of x such that —
$E \cap N = \phi$
$\Rightarrow N \subseteq E^c$
Hence, E^c is open

(1) —— Proved

Suppose E^c is open
let x be a limit point of E
Then, every deleted nbd of x contains atleast one point of E.
Thus, no nbd of x is contained in E^c.
Therefore, $x \notin Int(E^c)$
We know that E^c is open $\Rightarrow Int(E^c) = E^c$
$\Rightarrow x \notin E^c$
$\Rightarrow x \in E$
Hence, E is closed.

Theorem The intersection of an arbitrary collection of closed set is closed.

Proof → Let $\{E_i : i \in I\}$ where I is any index set, be an arbitrary collection of closed sets. & Put $E = \bigcap_{i \in I} E_i$

Using Demorgan's law, we have —

$$E^c = \left[\bigcap_{i \in I} E_i\right]^c = \bigcup_{i \in I} E_i^c$$

Since each E_i^c is open, therefore E^c is also open because it is the union of arbitrary collection of open sets.

Hence, E is closed.

—— Proved

Theorem The union of a finite collection of closed sets is closed.

Proof → Let $\{E_1, E_2, \ldots E_n\}$ be a finite collection of closed sets. and put $E = \bigcup_{i=1}^{n} E_i$

Using De-morgan's law ——

$$E^c = \left[\bigcup_{i=1}^{n} E_i\right]^c = \bigcap_{i=1}^{n} E_i^c$$

Since, each E_i^c is open; therefore E^c is open because it is the intersection of finite collection of open sets.

Hence, E is closed.

—— Proved