Theorem: If E is any subset of R then—
$$\overline{E} = E \cup D(E)$$

Proof → First, we show that $E \cup D(E)$ is closed.
Let 'x' be any limit point of $E \cup D(E)$
Then x must be a limit point of
E or $D(E)$.

Now, if x is a limit point of E
then $x \in D(E)$
Since, $D(E)$ is always closed. We have $x \in D(E)$

Thus, in both cases $x \in D(E)$. Hence, $x \in E \cup D(E)$
and consequently,
$E \cup D(E)$ is closed.

We now show that $\overline{E} = E \cup D(E)$.
Since $E \cup D(E)$ is closed superset of E and
$\overline{E}$ is the smallest closed superset of E.

$\Rightarrow \overline{E} \subseteq E \cup D(E)$ —— (1)

Next, Since $\overline{E}$ is closed we have $D(\overline{E}) \subseteq \overline{E}$.

Now,

$$E \subseteq \overline{E} \Rightarrow D(E) \subseteq D(\overline{E}) \subseteq \overline{E}$$

$$\Rightarrow D(E) \subseteq \overline{E}$$

Since,

$$E \subseteq \overline{E} \text{ and } D(E) \subseteq \overline{E}$$

$$\Rightarrow E \cup D(E) \subseteq \overline{E} \quad \text{——— } (2)$$

from eq (1) & (2), we get $\longrightarrow$

$$\overline{E} = E \cup D(E)$$

——— Proved