Theorem: The ordered field $\mathbb{Q}$ of rational numbers in not complete.

Proof: We have to show that the set $\mathbb{Q}$ doesn't satisfy completeness axiom.

For this, it supposes to produce a non-empty bounded above subset of $\mathbb{Q}$ which does not have a supremum in $\mathbb{Q}$.

let us consider, the set S of all those positive rational no's whose square is less than 2 i.e.

$$S = \{x : x \in \mathbb{Q}^+, x^2 < 2\}$$

Since $1 \in S \Rightarrow S$ is non-empty
Further, S is bounded above
2 being an upper bound of S.

We note that $2^2 = 4 > 2$ and so
$x \in \mathbb{Q}^+$ and $x^2 < 2 \Rightarrow x < 2$

Thus, S is an non-empty bounded above subset of $\mathbb{Q}$. let if possible, $\exists$ a real number α be a supremum of S.
Then $\alpha \geq 1$ and so $\alpha \in \mathbb{Q}^+$ Now one & only one of the following cases exists —

$\alpha^2 > 2$ $\qquad$ $\alpha^2 = 2$ $\qquad$ $\alpha^2 < 2$

The case $\alpha^2 = 2$ is not possible.
Since $\sqrt{2}$ is irrational.

~~The case α^2~~

Further if we define, $\beta = \dfrac{3\alpha+4}{2\alpha+3} \in \mathbb{Q}^+$ then.

$$\alpha - \beta = \alpha - \left(\frac{3\alpha+4}{2\alpha+3}\right)$$

$$\alpha - \beta = \frac{2(\alpha^2-2)}{2\alpha+3} \quad \text{①}$$

and

$$2-\beta^2 = 2-\left(\frac{3\alpha+4}{2\alpha+3}\right)^2$$

$$2-\beta^2 = \frac{2-\alpha^2}{(2\alpha+3)^2} \quad \text{②}$$

Now if $\alpha^2 > 2$ it follows from ① & ② that—

$\beta < \alpha$ and $\beta^2 > 2$

This shows that $\alpha \neq \operatorname{Sup} S$

Now if $\alpha^2 < 2$ it follows from ① & ② that—

$\alpha < \beta$ and $\beta^2 < 2$

This shows that $\alpha \neq \operatorname{Sup} S$

Summing up the above cases we conclude that the supremum of S does not exist in $\mathbb{Q}$.

Hence, $\mathbb{Q}$ is not complete.

Proved.