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Divergence Sequence:- A sequence of $\langle x_n \rangle$ is said to be divergent if —

$$\lim_{n\to\infty} x_n = \infty$$
or
$$\lim_{n\to\infty} x_n = -\infty$$

Eg ① $\left\langle \frac{n^2+3}{2n+3} \right\rangle \Rightarrow \lim_{n\to\infty} \frac{n^2+3}{2n+3} = \infty$

Hence, the given sequence diverges to ∞.

② $\left\langle \log \frac{1}{n} \right\rangle \Rightarrow \lim_{n\to\infty} \left(\log \frac{1}{n}\right)$

$= \log 0 = -\infty$

Hence the given sequence diverges to $-\infty$.

Oscillatory Sequence:- A sequence of $\langle x_n \rangle$ is said to be oscillatory sequence if it is neither convergent nor divergent.

Eg-① $\langle n(-1)^n \rangle \Rightarrow x_n = \begin{cases} -n, & n \text{ is odd} \\ n, & n \text{ is even} \end{cases}$

$\lim_{n\to\infty} x_n = \begin{cases} -\infty, & n \text{ is odd} \\ \infty, & n \text{ is even} \end{cases}$

Hence, the given sequence oscillates infinitely.

② $\left\langle (-1)^n n + \frac{1}{n} \right\rangle \rightarrow x_n$ $\quad x_n = \begin{cases} -n + \frac{1}{n}, & n \text{ is odd} \\ n + \frac{1}{n}, & n \text{ is even} \end{cases}$

$\lim_{n\to\infty} x_n = \begin{cases} -\infty, & n \text{ is odd} \\ \infty, & n \text{ is even} \end{cases}$

Hence, the given sequence oscillates infinitely

③ $\left\langle \sin \frac{n\pi}{2} \right\rangle \rightarrow x_n$

$\lim_{n\to\infty} x_n = \begin{cases} 1 \\ -1 \end{cases}$,

Hence the given sequence oscillates finitely b/w +1 and −1.

Theorem: If a sequence is convergent, then its limit is unique.

Proof: Let $\langle x_n \rangle$ be a convergent sequence. Let if possible it converges to two distinct limits l and l'.

$|x_n - l| < \epsilon \ \forall \ n \geq m$ and $|x_n - l'| < \epsilon \ \forall \ n \geq m'$

If $m_0 = \max\{m, m'\}$ where $\epsilon = \frac{1}{2}|l - l'|$

$|x_n - l| < \epsilon$ and $|x_n - l'| < \epsilon, \ \forall \ n \geq m_0$ — ①

Then, choosing $n \geq m_0$, we have —

$|l - l'| = |(x_n - l') - (x_n - l)|$

② $\left\langle (-1)^n n + \frac{1}{n} \right\rangle \to x_n$ $x_n = \begin{cases} -n + \frac{1}{n}, & n \text{ is odd} \\ n + \frac{1}{n}, & n \text{ is even} \end{cases}$

$\lim_{n\to\infty} x_n = \begin{cases} -\infty, & n \text{ is odd} \\ \infty, & n \text{ is even} \end{cases}$

Hence, the given sequence oscillates infinitely.

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Hence the given sequence oscillates finitely b/w $+1$ and -1.

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$|x_n - l| < \epsilon$ and $|x_n - l'| < \epsilon, \ \forall \ n \geq m_0$ —— ①

Then, choosing $n \geq m_0$, we have ——

$|l - l'| = |(x_n - l') - (x_n - l)|$

$|x-y| \geq |x| - |y|$

$|l-l'| \leq |x_n - l'| + |x_n - l|$

$|l-l'| < \epsilon + \epsilon$

$|l-l'| < 2\epsilon$

$|l-l'| < |l-l'|$

which is an absurd.

Hence, our assumption that $l \neq l'$ is wrong.

Therefore, the limit of a convergent sequence is unique.

——— Proved

Theorem: Every convergent sequence is bounded but the converge need not be true.

Proof → let $\langle x_n \rangle$ be a convergent sequence with 'l' as its limit.

Let us choose $\epsilon = 1$, then $\exists$ a positive integer 'm' such that ——

$$|x_n - l| < 1 \quad \forall\ n \geq m$$

$$\Rightarrow |x_n| - |l| < 1 \quad \forall\ n \geq m$$

$$\Rightarrow |x_n| < 1 + |l| \quad \forall\ n \geq m \quad \text{——— (1)}$$

If,

$$M = \max\{1+|l|, |x_1|, |x_2|, \ldots |x_{m-1}|\}$$

then, $|x_n| \leq M \quad \forall\ n \geq 1, 2, 3, \ldots (m-1)$

Hence, the sequence $\langle x_n \rangle$ is bounded.

$\langle (-1)^n \rangle$ is bounded but not convergent.

——— Proved