21/09/22

Monotone Convergence Theorem :-

ⓐ If a sequence $\langle x_n \rangle$ is monotonically increasing and bounded above, then it converges to its supremum.

ⓑ If a sequence $\langle x_n \rangle$ is monotonically decreasing and bounded below, then it converges to its infimum.

Proof - ⓐ let $E = \{x_n : n \in \mathbb{N}\}$ is a non-empty bounded above subset of $\mathbb{R}$.
Therefore, by the Completeness Property the supremum of E exists in $\mathbb{R}$.
(Therefore, $\exists$ atleast one term, say x_K in $\langle x_n \rangle$ such that —)

let $\sup E = L$. We have to show that $\langle x_n \rangle$ converges to L.
let $\epsilon > 0$ be a small +ve number then
$L - \epsilon < L$ and L is the supremum of E.
$\Rightarrow L - \epsilon$ cannot be an upper bound of E.

$$x_K > L - \epsilon \quad \text{———} ①$$

Further, since $\langle x_n \rangle$ is monotonic increasing for all $n \geq k$, we have —

$$x_K \leq x_n \quad \text{———} ②$$

Also, since L is the supremum of E, we have -

$$x_n \leq L \;\; \forall \; n \in \mathbb{N} \quad \text{———} ③$$

On combining eqⁿs (1), (2) & (3) we get →

$$L-\epsilon < x_k \leq x_n \leq L < L+\epsilon$$

$$\Rightarrow \quad L-\epsilon < x_n < L+\epsilon$$

$$\Rightarrow \quad |x_n - L| < \epsilon \quad \forall\ n \geq k$$

Hence, $\langle x_n \rangle$ converges to L.

—— Proved

(b) Proof →

let $E = \{x_n : n \in N\}$ is a non-empty bounded below subset of R.

Let $\inf E = l$ we have to show that $x_n \to l$

Let $\epsilon > 0$, $l + \epsilon > l$

Therefore, $\exists$ atleast one term say x_k in $\langle x_n \rangle$ such that ——

$$x_k < l + \epsilon \quad \text{—— (1)}$$

Further, since $\langle x_n \rangle$ is monotonically decreasing $\forall\ n \geq k$, we have ——

$$x_k \geq x_n \quad \text{—— (2)}$$

Also since l is a lower bound of E

$$x_n \geq l \quad \forall\ n \in N \quad \text{—— (3)}$$

On combining eq (1), (2) & (3), we get →

$$l-\epsilon < l \leq x_n \leq x_k < l+\epsilon$$

$$\Rightarrow \quad l-\epsilon < x_n < l+\epsilon \quad \forall\ n \geq k$$

$$\Rightarrow \quad |x_n - l| < \epsilon$$

Hence, $\langle x_n \rangle$ converges to l.

—— Proved