Q. Show that the sequence $\langle a_n \rangle$ where $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$ converges and its limit 'l' such that $\frac{1}{2} \le l \le 1$.

Solⁿ → Given → $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$

$$a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n+2}$$

$$a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} - \frac{1}{2n+2}$$

$$a_{n+1} - a_n = \frac{1}{(2n+1)(2n+2)} > 0$$

$\Rightarrow a_{n+1} > a_n \;\forall\; n \in N$

Therefore, the sequence $\langle a_n \rangle$ is monotonically increasing sequence.

Further, since the 1st term in the expansion of a_n is the greatest one, we have —

$$a_n < \frac{1}{n+1} + \frac{1}{n+1} + \dots + \frac{1}{n+1}$$

$$a_n < \frac{n}{n+1} < 1 \;\forall\; n \in N$$

$\Rightarrow \langle a_n \rangle$ is bounded above by 1.

Hence, by monotone convergence theorem $\langle a_n \rangle$ converges.

$$\Rightarrow \lim_{n\to\infty} a_n \le 1$$

$$a_n > \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}$$

$$a_n > \frac{n}{2n} > \frac{1}{2} \quad \forall \ n \in N$$

$$\Rightarrow \lim_{n\to\infty} a_n \ge \frac{1}{2}$$

Hence, $\frac{1}{2} \le \lim_{n\to\infty} a_n \le 1$

$$\frac{1}{2} \le l \le 1$$