$$a_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\cdots+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$$

Further for each $k=1,2,3,\ldots,(n-1)$ we have —

$$\frac{k}{n} > \frac{k}{n+1}$$

$$\Rightarrow \frac{-k}{n} < \frac{-k}{n+1}$$

$$\Rightarrow 1-\frac{k}{n} < 1-\frac{k}{n+1}$$

Therefore, each term from 3^{rd} to $(n+1)^{th}$ in the expansion a_n after the second is less than the corresponding terms of a_{n+1}.

$\Rightarrow a_n < a_{n+1}$

Hence, the given sequence is monotonic increasing.

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From ①, it is clear that —

$$a_n > 2 \quad \forall\ n \in N$$

Further from eq ①, we have —

$$a_n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$$

$$a_n < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$$

$$a_n < \cancel{1+} \frac{1-1/2^n}{1-1/2}$$

$$a_n < 1 + 2\left(1 - \frac{1}{2^n}\right)$$

$$a_n < 3 - \frac{1}{2^{n-1}}$$

$$a_n < 3$$

$$2 < a_n < 3 \quad \forall \; n \in \mathbb{N}$$

$$\Rightarrow 2 < \lim_{n \to \infty} a_n < 3$$

Hence, by monotonic convergence theorem the given sequence $\langle a_n \rangle$ is convergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left[1 + 1 + \frac{n(n-1)}{2!\, n^2} + \cdots\right]$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right)$$

$\lim_{n\to\infty} a_n = e$ (Unique limit)

Q. Show that the sequence →
$\sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, \ldots\ldots$ converges to 3.
$\underset{x_1}{\downarrow} \quad \underset{x_2}{\downarrow} \quad \underset{x_3}{\downarrow}$

$S_1 = \sqrt{3}$
$S_2 = \sqrt{3S_1}$
$S_3 = \sqrt{3S_2}$
$\vdots$
$S_{n+1} = \sqrt{3S_n}$

Clearly, $x_1 < x_2$. Also $x_k < x_{k+1}$
$\Rightarrow \sqrt{3x_k} < \sqrt{3x_{k+1}}$
$\Rightarrow x_{k+1} < x_{k+2}$
Thus by Mathematical Induction the given sequence is monotonic increasing.

Also $x_1 < 3$ and $x_k < 3$
$\Rightarrow \sqrt{3x_k} < \sqrt{3 \times 3}$
$\Rightarrow x_{k+1} < 3$
Again By Mathematical Induction,
$x_n < 3 \ \forall \ n \in \mathbb{N}$

By Monotonic Convergence Theorem, the given sequence is convergent.
Let $x_n \to l \Rightarrow \lim x_n = l = \lim x_{n+1}$
Now $\lim x_{n+1} = \lim \sqrt{3x_n}$
$\Rightarrow l = \sqrt{3l} \Rightarrow l = 0$ or 3
But $l \neq 0$, since $x_n \geq \sqrt{3} \ \forall \ n \in \mathbb{N}$
Hence $\lim_{n \to \infty} x_n = 3$