28/9/22

Q. Evaluate $\lim_{n\to\infty} \frac{n}{(n!)^{1/n}}$

By Cauchy second theorem on limits, we have-

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{a_n}{a_{n-1}}$$

$$\lim_{n\to\infty} \frac{n}{(n!)^{1/n}} = \lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{1/n}$$

$$a_n = \frac{n^n}{n!}$$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{\frac{n^n}{n!}}{\frac{(n-1)^{n-1}}{(n-1)!}}$$

$$= \lim_{n\to\infty} \frac{n^n (n-1)!}{n! (n-1)^{n-1}}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n-1)!}\,(n-1)}{n\cancel{(n-1)!}\left(1-\frac{1}{n}\right)^n}$$

$$= \lim_{n\to\infty} \frac{\left(1-\frac{1}{n}\right)}{e^{-1}}$$

$$= e \quad \underline{\text{Ans}}$$

Q If in a sequence $\langle a_n \rangle$ where $a_n = \frac{n!}{n^n}$

P.T → the sequence converges to 0.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}$$

$$= \lim_{n\to\infty} \frac{(n+1)! \cdot n^n}{n!\,(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)\,n!}{n!\,(n+1)\left(1+\frac{1}{n}\right)^n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{e} < 1$$

(We know that if $\frac{a_{n+1}}{a_n} \to l$ & $|l| < 1$
$\Rightarrow \lim_{n\to\infty} a_n = 0$)

Therefore,

$$\lim_{n\to\infty} a_n = 0$$

— Proved

Q. (i) If $\langle a_n \rangle$ where $a_n > 0\ \forall n$ and $\lim a_n = l > 0$ then prove that → $\lim\limits_{n\to\infty} (a_1 \cdot a_2 \cdots a_n)^{1/n} = l$

Let $x_n = \log a_n$ so that —

$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \log a_n = \log l$

Now, $\lim\limits_{n\to\infty} \dfrac{x_1 + x_2 + \cdots + x_n}{n} = \lim\limits_{n\to\infty} \dfrac{\log a_1 + \log a_2 + \cdots + \log a_n}{n}$

$= \lim\limits_{n\to\infty} \dfrac{1}{n} \log(a_1 \cdot a_2 \cdots a_n)$

$= \lim\limits_{n\to\infty} \log(a_1 \cdot a_2 \cdots a_n)^{1/n}$

$= \lim\limits_{n\to\infty} x_n = \log l$

$\Rightarrow \log \lim\limits_{n\to\infty} (a_1 \cdot a_2 \cdots a_n)^{1/n} = \log l$

$\lim\limits_{n\to\infty} (a_1 \cdot a_2 \cdots a_n)^{1/n} = l$

—— Proved

(ii) Find the limit of the sequence $\langle a_n \rangle$ where $a_n = (1 \cdot 2^{1/2} \cdot 3^{1/3} \cdots n^{1/n})^{1/n}$

$\lim\limits_{n\to\infty} (1 \cdot 2^{1/2} \cdot 3^{1/3} \cdots n^{1/n})^{1/n} = \lim\limits_{n\to\infty} n^{1/n} = 1$ Ans