Theorem: If $\langle y_n \rangle$ is a subsequence of a sequence $\langle x_n \rangle$ then —

(i) $\langle y_n \rangle$ is bounded if $\langle x_n \rangle$ is bounded.
(ii) $\langle y_n \rangle$ is monotonic if $\langle x_n \rangle$ is monotonic.
(iii) $\langle y_n \rangle$ is convergent if $\langle x_n \rangle$ is convergent.

Proof → Since $\langle y_n \rangle$ is a subsequence of $\langle x_n \rangle$ we have —

$y_n = x_{i_n}$ where $\langle i_n \rangle$ is a sequence of natural numbers such that $i_n < i_{n+1}$ and $i_n \geq n \ \forall \ n \in N$.

(i) If $\langle x_n \rangle$ is bounded, then $\exists$ real numbers 'm' and 'M' such that —

$$m \leq x_n \leq M \ \forall \ n \in N$$

In particular, we have —

$$m \leq x_{i_n} \leq M \ \forall \ n \in N$$

Hence, $\langle y_n \rangle$ is bounded.

— Proved

(ii) If $\langle x_n \rangle$ is monotonic increasing, then —

$i_n < i_{n+1} \Rightarrow x_{i_n} \leq x_{i_{n+1}} \Rightarrow y_n \leq y_{n+1} \ \forall \ n \in N$

Hence, $\langle y_n \rangle$ is monotonic increasing.

If $\langle x_n \rangle$ is monotonic decreasing, then —

$i_n < i_{n+1} \Rightarrow x_{i_n} \geq x_{i_{n+1}} \Rightarrow y_n \geq y_{n+1} \ \forall \ n \in N$

Hence, $\langle y_n \rangle$ is monotonic decreasing.

— Proved

(iii) Let $\epsilon > 0$ be given. If $\langle x_n \rangle$ converges to l, then $\exists$ a positive number 'm' such that —

$n \geq m \Rightarrow |x_n - l| < \epsilon$

Since, $i_n \geq n$, we have —

$n \geq m \Rightarrow i_n \geq m \Rightarrow |x_{i_n} - l| < \epsilon$

$\Rightarrow |y_n - l| < \epsilon$

Hence, $\langle y_n \rangle$ converges to l.

— Proved

Corollary:- A sequence of $\langle x_n \rangle$ converges to l iff $\lim x_{2n} = \lim x_{2n+1} = l$

Proof → If $\langle x_n \rangle$ converges to l, then by previous result (iii) its subsequence also converges to l

Hence, $\lim x_{2n} = \lim x_{2n+1} = l$

Conversely,

If $\langle x_{2n} \rangle \to l$ and $\langle x_{2n+1} \rangle \to l$

$\Rightarrow |x_{2n} - l| < \epsilon$ and $|x_{2n+1} - l| < \epsilon$

and so

$n \geq m \Rightarrow |x_n - l| < \epsilon$

Hence, $\langle x_n \rangle$ converges to l.