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Bolzano-Weirestrass Theorem for sequence

"Every bounded sequence has a convergent subsequence."

Proof → Let E be the set of all distinct points of a bounded sequence $\langle x_n \rangle$.
Then E is also bounded.

(#) If E is finite there must be atleast one element say 's' $\in$ E which is infinitely repeated in $\langle x_n \rangle$.
Let $\langle i_n \rangle$ be increasing sequence of positive integers such that —

$$x_{i_n} = s \ \forall \ n \in N$$

Obviously $\langle x_{i_n} \rangle$ is a subsequence of $\langle x_n \rangle$ and being a constant sequence.
Hence, it converges to 's'.

(#) If E is infinite, then by Bolzano-Weirestrass Theorem for sets, it has a limit point say 'x' in R.
We shall now construct a subsequence of $\langle x_n \rangle$ which converges to x.
For each $K \in N$ let $I_K = \left(x - \frac{1}{k}, x + \frac{1}{k}\right)$ be the $(1/k)$-nbd of x.
Since, x is a limit point of E,
the set I_K contains infinitely many values ~~of 'π' such~~ element of E.

Thus, for each k there are infinitely many values of 'n' such that —

$$x_n \in I_n$$

Now, choose $x_{i_1} \in I_1$, $x_{i_2} \in I_2$, $x_{i_3} \in I_3$ and so on such that $i_3 > i_2 > i_1$

Thus, we obtain a subsequence $\langle x_{i_n} \rangle$ of $\langle x_n \rangle$ such that — $x_{i_n} \in I_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$

i.e. $|x_{i_n} - x| < \frac{1}{n} \ \forall \ n \in N$

Hence $\lim x_{i_n} = x$

——— Proved

Q Show that the sequence $\langle (-1)^{n-1} \rangle$ is not convergent.

let $\langle x_n \rangle$ be a sequence and let $\langle x_{2n-1} \rangle$ and $\langle x_{2n} \rangle$ are two subsequences of $\langle x_n \rangle$

$\langle x_{2n-1} \rangle = \langle 1, 1, 1, \ldots \rangle$
$\langle x_{2n} \rangle = \langle -1, -1, -1, \ldots \rangle$

$\lim x_{2n-1} = 1$ and $\lim x_{2n} = -1$

Since, the two subsequences of the given sequence converges to 2 different limits.
Therefore, $\langle (-1)^{n-1} \rangle$ is not convergent.